Large time decay of the heat kernel of $\lambda$–transient Riemannian manifolds

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Let $M$ be a noncompact Riemannian manifold with Laplace–Beltrami operator $\Delta$ acting on functions on $M$, $\lambda =: \lambda(M) \geq 0$ the bottom of spec $(-\Delta)$, and $p(x, y, t)$ (where $(x, y, t)$ is an element of $M \times M \times (0, +\infty)$) the attendant minimal positive heat kernel. In this note we prove the following

**Theorem.** Assume that

\begin{equation}
\int_0^{+\infty} e^{\lambda t} p(x, y, t) \, dt < +\infty
\end{equation}

for some pair $(x, y)$, $x \neq y$. Then

\begin{equation}
e^{\lambda t} p(x, y, t) = o(t^{-1})
\end{equation}

as $t \uparrow +\infty$, uniformly on compact subsets of $M \times M$.

The inequality (1) is usually referred to as $\lambda$–transience of $M$. See [5]; also see [3]. We add that in [3] it is shown that, in general, $e^{\lambda t} p$ always has a limit when $t \uparrow +\infty$, which either is identically equal to 0 on all of $M \times M$ or never vanishes on all of $M \times M$. So our discussion here is within the context of the first of the two alternatives.

The result (2) is best possible in the sense that in $\mathbb{R}^2$ one has $\lambda = 0$, $p(x, x, t) = 1/4\pi t$, and the integral in (1) diverges.

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The result is sharp, in the sense that that one can construct (see Benjamini [1]) a surface of revolution which is 0–transient and whose heat kernel satisfies
\[ p(o, o, t) \geq \frac{\text{const.}}{t \ln^2 t} \]
where \( o \) is the pole of the surface, for large \( t \). (See Benjamini’s paper for other examples.)

We note that if \( M \) is complete noncompact with bounded geometry (that is, Ricci curvature bounded from below and strictly positive injectivity radius), then it is a general result of Chavel–Feldman [2], improving an earlier result of Varopoulos [6], that
\[ p(x, y, t) = O(t^{-1/2}) \]
as \( t \uparrow +\infty \).

We also note that the semigroup property and the Cauchy–Schwarz inequality imply
\[ p(x, y, t) \leq \sqrt{p(x, x, t)} \sqrt{p(y, y, t)}; \]
so it suffices to prove (2) for \( x = y \).

**Lemma.** For all \( x \in M \) we have
\[ e^{\lambda t} p(x, x, t) \]
is a decreasing function of \( t \).

**Proof.** Let \( D \) be a relatively compact domain in \( M \) with smooth boundary, and Dirichlet heat kernel \( q \). Then one has \( e^{\lambda t} q(x, x, t) \) is a decreasing function of \( t \) from the Sturm–Liouville eigenvalue–eigenfunction expansion of \( q \).

Now pick an exhaustion of \( M, D_j \uparrow M \) as \( j \uparrow +\infty \), by domains which are relatively compact in \( M \) and which possess smooth boundary. Let \( q_j \) denote the Dirichlet heat kernel of \( D_j \). It is standard that
\[ q_j \uparrow p. \]
The lemma follows immediately. \( \Box \)

The lemma seems to have been known for some time. Peter Li has shown us (private communication — in which he derives the lemma with a cut–off function argument instead of our argument) that one can easily deduce from the lemma and from the monotonicity of \( \lambda_1(D) \) with respect to \( D \) (\( \lambda_1(D) \) is the lowest eigenvalue of \( D \) for \( D \) relatively compact) that
\[ \lim_{t \uparrow +\infty} \frac{\ln p(x, y, t)}{t} = -\lambda. \]
See [4]. For recent improvements see [3].
Proof of the Theorem. It is standard (see [5]) that if (1) is valid for a given pair 
\((x, y)\), \(x \neq y\), then (1) is valid for all \(x \neq y\). Furthermore, if for any fixed \(x\) we set

\[
D_x(y) = \int_1^{+\infty} e^{\lambda t} p(x, y, t) \, dt \quad y \in M,
\]

then \(D_x \in L_{\text{loc}}^1(M)\).

Now consider \(\Delta D_x\) as a distribution. Given \(\varphi \in C_c^\infty(M)\) we have

\[
(\Delta D_x) \cdot \varphi = \int D_x(y)(\Delta \varphi)(y) \, dV(y)
\]

\[
= \lim_{T \uparrow +\infty} \int_1^T e^{\lambda t} dt \int_M p(x, y, t)(\Delta \varphi)(y) \, dV(y)
\]

\[
= \lim_{T \uparrow +\infty} \int_1^T e^{\lambda t} dt \int_M (\Delta y p)(x, y, t) \varphi(y) \, dV(y)
\]

\[
= \lim_{T \uparrow +\infty} \left\{ \int_M e^{\lambda T} p(x, y, T) \varphi(y) \, dV(y) - \int_M e^\lambda p(x, y, 1) \varphi(y) \, dV(y)
- \lambda \int_1^T e^{\lambda t} dt \int_M p(x, y, t) \varphi(y) \, dV(y) \right\}
\]

\[
= -\lambda D_x \cdot \varphi - \int_M e^\lambda p(x, y, 1) \varphi(y) \, dV(y),
\]

by the \(\lambda\)-transience of \(M\) (see (5.5) of [5]). Therefore,

\[
\Delta D_x + \lambda D_x = e^\lambda p(x, , 1)
\]

(3)

in the sense of distributions. But the right hand side of (3) is \(C^\infty\) on all of \(M\). Thus the distribution \(D_x\) may be redefined on a set of measure 0 to become a \(C^\infty\) function \(K\) on \(M\).

Now approach \(x\) along a sequence of points for which \(D_x\) and \(K\) agree. Since the values of \(K\) are bounded on the sequence, one concludes via Fatou’s lemma that

\[
\int_1^{+\infty} e^{\lambda t} p(x, x, t) \, dt < +\infty.
\]

Since \(e^{\lambda t} p(x, x, t)\) decreases with respect to \(t\), (2) follows.

The uniformity of the convergence on compact subsets also follows from the monotonicity of \(e^{\lambda t} p(x, x, t)\). \(\square\)

References


