CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
FALL 2015
3 hours

Instructions. The exam consists of two parts. Choose a *total of seven problems*, including *at least three from each part.* Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. Let G be the group with presentation $\langle x, y \mid yx^3y^3x^{-3}y^{-4}\rangle$. Prove that the image in G of xy has infinite order.

2. Prove that there is no simple group of order 132.

3. Let $n \geq 3$ be an integer. Let $V_1, V_2, \ldots, V_n$ be vector spaces over a field F such that the sequence of maps $0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$ is exact. If $m_i = \dim_F V_i$, show that $\sum_{i=1}^{n}(-1)^{i+1}m_i = 0$.

4. Let K be a field and A a commutative K-algebra, finite dimensional as a vector space over K. Prove that every prime ideal of A is maximal.

5. Let R be a commutative ring with unity and let P, Q be projective R-modules. Prove that $P \otimes_R Q$ is a projective R-module.

Part II

6. Prove or disprove that the $\mathbb{Q}$-algebra $\mathbb{Q}(\sqrt[3]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_3)$ is a field (here ζ_3 denotes a primitive third root of unity). Similarly, prove or disprove that the $\mathbb{R}$-algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a field.

7. Let $f(x) = x^4 - 5x^2 + 5$ in $\mathbb{Q}[x]$ and $K = \mathbb{Q}(\alpha)$, where α is a root of f in an algebraic closure of $\mathbb{Q}$.
   a. Find the degree $[K:\mathbb{Q}]$.
   b. Show that $\sqrt{5}$ is in K and that $\beta = \dfrac{\sqrt{5}}{\alpha}$ also is a root of f.
   c. Prove that $K/\mathbb{Q}$ is a Galois extension and determine the structure of $\mathrm{Gal}(K/\mathbb{Q})$.

8. For prime p, let F be the splitting field of $f(x) = x^5 - p$ over $\mathbb{Q}$.
   a. Determine the structure of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$.
   b. How many subfields E of F have degree 5 over $\mathbb{Q}$? Justify your answer.

9. Let $\mathbb{F} = \mathbb{F}_q$ be the field with $q = p^m$ elements for a prime p and let $g(x) = x^d - 1$ in $\mathbb{F}[x]$.
   a. Show that g splits completely into linear factors over $\mathbb{F}$ if and only if $q \equiv 1 \pmod{d}$.
   b. Let K be the splitting field of $f(x) = x^5 - 2$ over $\mathbb{F} = \mathbb{F}_{11}$. Determine the structure of $\mathrm{Gal}(K/\mathbb{F}_{11})$.

(continued on page 2)

10. Using the standard bases, let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   a. Find the characteristic polynomial of T and the eigenvalues of T.
   (As a check on your work, note that all the eigenvalues are real.)

   b. For each eigenvalue λ, find a basis for the λ-eigenspace.

   c. Determine the Jordan form and minimal polynomial of T.

11. Prove the Hilbert basis theorem: a polynomial ring $R[x]$ over a Noetherian ring R is Noetherian.
   Suggestion. Let A be a non-trivial ideal in $R[x]$ and use the ideals I_d in R consisting of 0 and the leading coefficients of all polynomials in A of degree d.