CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
FALL 2017
3 hours

Instructions. The exam consists of two parts. Choose a *total of seven problems*, including *at least three from each part*. Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. Prove that no group of order $1000000 = 2^6 \cdot 5^6$ is simple.

2. Let R be a ring and let $M = M_1 \oplus M_2$ be a direct sum of R-modules M_1, M_2. Give an example of a R-submodule N of M such that N is not the direct sum of $N \cap M_1$ and $N \cap M_2$.

3. Let A be an $n \times n$ matrix with entries in a field K, let $f \in K[x]$ be a polynomial with coefficients in K and let $B = f(A)$.
   a. Let v be an eigenvector of A with eigenvalue λ. Show that v also is an eigenvector of B. What is the corresponding eigenvalue?
   b. Assume that B is invertible. Prove that there is a polynomial $g \in K[x]$ such that $B^{-1} = g(A)$.

4. List all possible rational canonical forms of 6×6 real matrices M satisfying $(M^2 - M + 1)^3 = 0$.

5. Express the following $\mathbb{Q}[x]$-module as a direct sum of cyclic $\mathbb{Q}[x]$-modules:
$$\mathbb{Q}[x]/((x-2)(x^2+1)^2) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/((x-2)^2(x^2+1)).$$

Part II

6a. Let S_4 be the symmetric group on 4 letters. Prove that all subgroups of order 8 in S_4 are isomorphic to the dihedral group D_4.
 b. Prove that the Galois group over $\mathbb{Q}$ of $x^4 - 5$ is isomorphic to D_4.

7a. Let F be a field whose characteristic does not divide n. Prove that $f(x) = x^n - 1$ is separable over F and that the Galois group of f is isomorphic to a *subgroup* of $(\mathbb{Z}/n\mathbb{Z})^\times$.
 b. Construct a field F such that $p(x) = x^{15} - 1$ is separable over F and the Galois group of $p(x)$ is not isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$.

8. Let V be an n-dimensional vector space over the p-element field $\mathbb{F}_p$ and let G be a finite p-group of $n \times n$ matrices over $\mathbb{F}_p$. Prove that there is a non-zero $v \in V$ such that $g(v) = v$ for all $g \in G$.

9. Let $G = \langle a, b \rangle$ be an infinite group generated by two elements a, b, each of order two. Prove that G is isomorphic to the group of permutations of $\mathbb{Z}$ generated by $f(x) = -x$ and $g(x) = 1 - x$.
   Hint: Show each element of G can be uniquely written as an alternating product of a's and b's.

10. Let p and q be distinct primes and let G be a *non-abelian* group of order pq. Prove that G is solvable, but not nilpotent.

Reminder: please indicate on the cover of your answer book the seven problems you have chosen for grading.