CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
FALL 2018
3 hours

Instructions. The exam consists of two parts. Choose a *total of seven problems*, including *at least three from each part*. Indicate on the first page of your exam the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. Let R be a commutative ring with identity and suppose that $f_1, \ldots, f_n \in R$ are elements such that there exist $g_i \in R$ with $\sum g_i f_i = 1$. Let M be an R-module.
   (a) Show that $M = 0$ if and only if for all i, the R_{f_i}-module $M_{f_i} = 0$.
   (b) Show that M is finitely generated if and only if for all i, the R_{f_i}-module M_{f_i} is finitely generated.
2. Let R be an integral domain and M an R-module. Denote by $\tau(M)$ the torsion submodule of M.
   (a) Show that if
   $$0 \longrightarrow M' \longrightarrow M \longrightarrow M''$$
   is exact, then
   $$0 \longrightarrow \tau(M') \longrightarrow \tau(M) \longrightarrow \tau(M'')$$
   is exact.
   (b) Is $M \mapsto \tau(M)$ an exact functor? Give a proof or counterexample.
3. Prove that there is a unique nonabelian group G of order 253 up to isomorphism. Prove also that G is not simple.
4. Express the following $\mathbb{Q}[x]$-module as direct sum of cyclic $\mathbb{Q}[x]$-modules:
   $$\mathbb{Q}[x]/((x-2)(x^2+1)^2) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/((x-2)^2(x^2+1))$$
5. Show that the nonprincipal ideal $J = (3, 2 + i\sqrt{5})$ is a projective module in the ring $R = \mathbb{Z}[\sqrt{-5}]$.
6. Let G be a finite nilpotent group. Prove that for every prime p, G has a unique p-Sylow subgroup.

Part II

7. Let $f(x) = x^n - 1$ in $F[x]$. Let K be the splitting field of f over F. Prove that K is a Galois extension of F and that $\mathrm{Gal}(K/F)$ is abelian.
8. Prove that an element g of a finite group G is conjugate to g^{-1} if and only if $\chi(g)$ is a real number for every character χ of G.
9. Let K be the splitting field of the polynomial $x^4 + 1$.
   (a) Compute $\mathrm{Gal}(K/\mathbb{Q})$.
   (b) How many fields F are there with $\mathbb{Q} \subseteq F \subseteq K$?
10. Compute the Galois group over $\mathbb{Q}$ of $x^6 - 10x^3 + 1$.
11. Let F be a field and $F(\alpha)$ a simple algebraic extension. Suppose that the minimal polynomial of α over F is $f(x)$. Let K be an extension field of F. Prove that the number of F-algebra homomorphisms from $F(\alpha)$ to K is finite and equal to the number of roots of $f(x)$ in K.
12. Let I, J be two ideals of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Assume that IJ is a radical ideal. Prove that $IJ = I \cap J$.

Reminder: please indicate on the first page of your exam the seven problems you have chosen for grading.