# CUNY GRADUATE CENTER
# DEPARTMENT OF MATHEMATICS
# ALGEBRA QUALIFYING EXAM
# FALL 2020

**Instructions:** The exam consists of two parts. Choose a total of *seven problems*, including at least three from each part. Indicate on the first page of your exam the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

**Notation:** In this exam $\mathbf{Z}$ stands for the ring of rational integers, and $\mathbf{Q}$ for the field of rational numbers.

## Part 1

(1) This question has two parts:
  (a) Let $A$ be a nonempty subset of a group $G$. Prove that $A$ is a subgroup if and only if $xA = A$ for all $x \in A$.
  (b) Let $G$ be a group, and let $a$ and $b$ be elements of $G$ such that $a^{-1}ba = b^2$ and $b^{-1}ab = a^2$. Prove that $a = b = e$. Here $e$ denotes the identity in $G$.

(2) Let $g(x) = x^3 - 9x - 9$. Recall that the discriminant of a cubic polynomial $x^3 - ax - b$ is given by $4a^3 - 27b^2$.
  (a) Show that $g$ is irreducible over $\mathbf{Q}$.
  (b) Let $F$ be the splitting field of $g$ over $\mathbf{Q}$. Determine the Galois group $\mathrm{Gal}(F/\mathbf{Q})$.

(3) Let $n \geq 5$. Write $S_n$ for the symmetric group on $n$ letters and $A_n$ for the alternating group. Let $\sigma \in S_n$. Prove that $\sigma \in A_n$ if and only if $\sigma$ is a product of 3-cycles.

(4) Let $F_2 = \langle a, b \rangle$ be the free group of rank 2. Prove that the subgroup of $F_2$ generated by the set $\{a^n b a^{-n} : n = 1, 2, 3, ...\}$ is a free group of countably infinite rank.

(5) Prove that $\sqrt{1 + \sqrt[5]{3}}$ is not a constructible number.

(6) Let $R = \mathbf{Z}[\sqrt{-13}]$ and consider the ideal $A = (2, 1 - \sqrt{-13})$ in $R$. Show that $A$ is maximal but not principal.

# PART 2

(1) Let $R$ be a commutative ring with 1. Let $I$ be a principal ideal of $R$. Prove that $I$ is a free $R$-module if and only if it is generated by an element of $R$ which is not a zero-divisior. Demonstrate by an example that if $I$ is not principal it need not be a free $R$-module even if $I$ is generated by non-zero-divisors.

(2) Let $F$ be a field and $G$ a finite abelian group such that $\mathrm{char}(F) \nmid |G|$. Let $V$ be an $F[G]$-module. Let $F(\chi)$ denote a one-dimensional $F$-vector space on which $G$ acts via a homomorphism $\chi : G \to F^\times$, i.e., $g \cdot \alpha = \chi(g)\alpha$ for all $g \in G$, $\alpha \in F(\chi)$. Set $V^\chi := \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in G\}$. Prove that

$$V \otimes_{F[G]} F(\chi) \cong V^\chi \quad \text{as } F[G]\text{-modules.}$$

Hint: Define $e_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g)^{-1} g \in F[G]$.

(3) Find representatives for all the different similarity classes of $5 \times 5$ matrices with rational coefficients whose minimal polynomial is $(x-1)^3(x-2)$

(4) Let $V$ be a vector space of dimension $n$ over a field $K$. We say that an endomorphism $T : V \to V$ is nilpotent if there is a positive integer $m$ such that $T^m = 0$.
   (a) Let $T : V \to V$ be any endomorphism of $V$ and let $V_0 = \ker T$. Show that there exists a linear subspace $W$ of $V$ such that $V = V_0 + W$ and $V_0 \cap W = \{0\}$. Describe the matrix representation of $T$ with respect to this decomposition of $V$.
   (b) Suppose that $T$ is nilpotent. Show that the trace $\mathrm{tr}\,(T) = 0$.
   (c) Suppose that $K$ has characteristic 0 and suppose instead that $T$ is an endomorphism satisfying $\mathrm{tr}\, T^r = 0$ for all $r \geq 1$. Prove that $T$ is nilpotent. **Suggestion:** Show that $\ker T$ is not trivial and use part (a) to facilitate an argument by induction on $\dim V$ .

(5) Show that every module over the ring $\mathbf{Z}/15\mathbf{Z}$ is projective.

(6) Let $R$ be a commutative ring with 1. For an $R$-module $L$ we write $L^\vee$ for $\mathrm{Hom}_R(L, R)$. Let $M$ and $N$ be free $R$-modules of finite rank.
   (a) Show that $N \otimes_R M^\vee \cong \mathrm{Hom}_R(M, N)$.
   (b) If $M = M_1 \oplus M_2$ for some $R$-submodules $M_1$ and $M_2$ of $M$, we write $M_1' \subset M^\vee$ for the submodule consisting of maps that annihilate $M_1$. Show that $N \otimes_R M_1'$ is isomorphic to the submodule of $\mathrm{Hom}_R(M, N)$ consisting of maps that annihilate $M_1$.