In Part I:

1. Let \( f(x) = x^3 - 5x^2 + 10x + 20 \) in \( \mathbb{Z}[x] \) and let \( (f) = f\mathbb{Z}[x] \) be the principal ideal generated by \( f \). Let \( \mathbb{Z}[\sqrt{3}] \) be the quadratic integer ring \( \{a + b\sqrt{3} | a, b \in \mathbb{Z}\} \). Let \( R \) be the ring \( \mathbb{Z}[\sqrt{3}] \otimes_{\mathbb{Z}} (\mathbb{Z}[x]/f\mathbb{Z}[x]) \). Prove or disprove that \( R \) is an integral domain.

2. Let \( p \) and \( q \) be distinct primes, \( R = \mathbb{Z}/pq\mathbb{Z}, x \) an indeterminate over \( R \) and \( S \) the ring \( R[x]/x^3R[x] \). Determine:
   a. all the maximal ideals of \( S \);
   b. all the prime ideals of \( S \);
   c. the nilradical of \( S \).

3. Let \( G \) be the group \( \mathbb{Z}/35\mathbb{Z} \) and \( H \) the group \( \mathbb{Z}/15\mathbb{Z} \).
   a. Determine all isomorphism classes of semidirect product groups \( G \rtimes_{\varphi} H \), where \( \varphi \) is a homomorphism from \( H \) to \( \text{Aut} \, G \).
   b. With \( G \) and \( H \) as above, give an example of a group \( K \) such that the following sequence is exact, but does not split: \( 0 \to G \to K \to H \to 0 \).

4. Prove that a group of order 132 is not simple.

5. For each of the following, either construct the object or prove it does not exist:
   a. an integral domain consisting of 15 elements;
   b. an integral domain with non-zero characteristic and infinitely many elements.

6. Let \( R \) be the quadratic integer ring \( R := \{a + b\sqrt{-2} | a, b \in \mathbb{Z}\} \). Prove that \( R \) is an Euclidean domain with respect to the norm \( N(a + b\sqrt{-2}) = (a + b\sqrt{-2})(a - b\sqrt{-2}) \).

(see next page for Part II)
Part II

7. Let $V$ be a finite dimensional vector space over $\mathbb{Q}$ and let $T: V \to V$ be a linear transformation such that $T^2 = -1$. Suppose $V$ has a non-trivial proper subspace $W$ invariant under $T$. What is the smallest possible value for $\dim_{\mathbb{Q}} V$?

8. Let $K$ be a finite extension field of $F$. Prove that $K$ is a splitting field over $F$ if and only if every irreducible polynomial in $F[x]$ that has a root in $K$ splits completely in $K$.

9. Let $\mathbb{F}_p$ denote the field with $p$ elements. Determine the Galois group of the splitting field of $x^3 - x + 1$ over the following fields: 
a) $\mathbb{F}_3$  
b) $\mathbb{F}_5$  
c) $\mathbb{Q}$.
(Recall that the discriminant $D(f)$ of $f(x) = x^3 + bx + c$ is given by $D(f) = -4b^3 - 27c^2$.)

10. Let $K$ be a finite extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ a root of $f(x) = x^6 + 3$.
   a. Show that $K$ contains a primitive sixth root of unity.
   b. Show that $K$ is Galois over $\mathbb{Q}$.
   c. Determine the number of fields $F$ of degree 3 over $\mathbb{Q}$ and contained in $K$.

11. Let $K/F$ be a finite extension of finite fields. Prove that the norm map $N_{K/F}: K \to F$ is surjective.

12. Let $k$ be an algebraically closed field and $A^n = k^n$ denote the affine space of dimension $n$. Consider the zero set 
$$Y = \{x^2 - yz, xz - x\} \subset A^3.$$ 
Decompose $Y$ into irreducible components.