CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
SPRING 2016
3 hours

**Instructions.** The exam consists of two parts. Choose a *total of seven problems*, including *at least three from each part.* Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

## Part I

1. Let $G$ be a finite group of order $616 = 2^3 \cdot 7 \cdot 11$.
   a. Show that G is not simple.
   b. Show that G has a subgroup of order 77 or 88.

2. Let $G$ be a finite group and let $H \leq G$ be a subgroup of index $[G\colon H] = n$. Write $H^g = g^{-1}Hg$ for conjugation by $g \in G$.
   a. Show that $[H\colon (H \cap H^g)] \leq n$ for all $g \in G$.
   b. If H is a maximal proper subgroup of $G$ and $H$ is abelian, show that $H \cap H^g$ is a normal subgroup of $G$ for all $g \notin H$.
   c. Now suppose that $G$ is simple. If $H$ is abelian and $n$ is a prime, prove that $H = \{1\}$.

3. Let $H$ be the subgroup of the free group $\mathcal{F}(a, b)$ given by $H = \langle a^2, b^2, aba^{-1}, bab^{-1}\rangle$.
   a. Is the index of $H$ in $\mathcal{F}(a, b)$ finite or infinite?
   b. Does $abababa$ belong to $H$?

4. Let $K$ be a field and let $R$ be the subring of the polynomial ring $K[x]$ given by all polynomials with $x$-coefficient equal to 0.
   a. Prove that the elements $x^2$ and $x^3$ are irreducible but not prime in the ring $R$.
   b. Show that $R$ is a Noetherian ring and that the ideal $I$ of $R$ consisting of all polynomials in $R$ with constant term 0 is not principal.

5. Prove that each of the following polynomials is irreducible in the indicated polynomial ring.
   a. $f(x, y) = x^3 - xy^2 - 7x - 2y^2 + 14$ in $\mathbb{Q}[x, y]$.
   b. $g(x) = x^{100} - 100x - 10$ in $F[x]$, where $F = \mathbb{Q}[\sqrt{-2}]$.
   c. $h(x) = 5x^5 - 4x^3 - 3x^2 - 2x - 1$ in $\mathbb{Z}[x]$.

## Part II

6. Let $f(x) = x^4 + 14x^2 - 7x + 28$ in $\mathbb{Z}[x]$.
   a. Show that the Galois group $\mathrm{Gal}_{\mathbb{Q}}(f)$ of $f$ over $\mathbb{Q}$ is the symmetric group $S_4$ on 4 letters.
   b. Let $K$ be the splitting field of $f$ over $\mathbb{Q}$. Determine the number of different intermediate fields $F$ such that $K \supset F \supset \mathbb{Q}$ and $[F\colon \mathbb{Q}] = 3$.

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7. Let $f$ be an *irreducible cubic* polynomial over a field $F$ with $\mathrm{char}(F) \neq 3$. Justify each of the following statements.
   a. $f$ is a separable polynomial; i.e. $f$ has no repeated roots in its splitting field.
   b. The Galois group $\mathrm{Gal}_F(f)$ is the alternating group $\mathcal{A}_3$ or the symmetric group $\mathcal{S}_3$.
   c. If, in addition, $F$ is a finite field, then the discriminant of $f$ is a square in $F$.

8. Let $R$ be a ring with 1, let $D$ be a left $R$-module and let $0 \to M_1 \xrightarrow{i} M \xrightarrow{j} M_2 \to 0$ be an exact sequence of left $R$-modules.
   a. State the definitions of the induced maps $j^*$ and $i^*$ in
   $$\to \mathrm{Hom}_R(M_2, D) \xrightarrow{j^*} \mathrm{Hom}_R(M, D) \xrightarrow{i^*} \mathrm{Hom}_R(M_1, D) \to$$
   b. Prove exactness around $\mathrm{Hom}_R(M, D)$, namely, $\ker i^* = \text{image } j^*$.

9. Let $I$ and $J$ be ideals in a commutative ring $R$ with 1.
   a. Show that every element of $(R/I) \otimes_R (R/J)$ has the form $\overline{r} \otimes \overline{1}$, where $\overline{r}$ ranges over $R/I$ and $\overline{1}$ is the image of 1 in $R/J$.
   b. Find an explicit isomorphism of $R$-modules $(R/I) \otimes_R (R/J) \xrightarrow{\sim} R/(I+J)$.

10. The matrix below represents a linear transformation $T\colon \mathbb{C}^4 \to \mathbb{C}^4$ whose characteristic polynomial is $p(x) = (x-2)^4$.
   $$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}.$$
   a. Find the Jordan canonical form for $T$.
   b. Find the minimal polynomial for $T$.

Reminder: please indicate on the cover of your answer booklet the 7 problems you have chosen for grading.