CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
SPRING 2017
3 hours

Instructions. The exam consists of two parts. Choose a *total of seven problems*, including *at least three from each part*. Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. Prove that no group of order 3000 is simple.

2. a. Describe all $\mathbb{Z}$-module homomorphisms $f\colon \mathbb{Z}/15\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$.
   b. Let $A = \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$ and $B = 3\mathbb{Z}$.
      i) Find the annihilator $\mathrm{Ann}_{\mathbb{Z}}(A)$ of A in $\mathbb{Z}$.
      ii) Describe the set $S = \{a \in A \mid ba = 0 \text{ for all } b \in B\}$ as a direct product of cyclic groups.

3. Let R be an integral domain. Prove that every R-module is projective if and only if R is a field.

4. Provide an example of a non-trivial abelian group G such that $G \otimes_{\mathbb{Z}} G = 0$ and verify your example.

5. Let $R = \mathbb{Q}[x]$ and let M be the submodule of $R \oplus R$ spanned by $(2x+3, x-1)$ and $(3x+5, x+1)$. Determine the structure of the quotient module $(R \oplus R)/M$ as a direct sum of cyclic R-modules.

6. Let R be a commutative ring with 1, let C be the category of R-modules and let M be an R-module.
   a. Show that the functor $- \otimes_R M$ from C to C is right exact.
   b. Give an example of M for which this functor is not left exact.

Part II

7. Let p be a prime.
   a. Prove that the symmetric group S_p is generated by any transposition and p-cycle.
   b. Prove that if $f(x) \in \mathbb{Q}[x]$ is irreducible of degree p with exactly $p-2$ real roots, then the Galois group of f is isomorphic to S_p.

8. Let $f(x) = x^4 + 30x^2 + 45 \in \mathbb{Q}[x]$ and let $\alpha = \sqrt{-15 + 6\sqrt{5}}$.
   a. Prove that $f(x)$ is irreducible.
   b. Prove that $\mathbb{Q}(\alpha)$ is a splitting field for $f(x)$.
      Hint: show $\beta = \sqrt{-15 - 6\sqrt{5}}$ is in $\mathbb{Q}(\alpha)$ by showing that $\alpha\beta \in \mathbb{Q}(\sqrt{5})$.
   c. Prove that $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ is cyclic of order 4.
   d. Give a complete list of intermediate fields K with $\mathbb{Q} \subseteq K \subseteq \mathbb{Q}(\alpha)$.

9. Let $L = \mathbb{C}(x_1, x_2, x_3)$ be the field of rational functions in three variables over $\mathbb{C}$ and let K be the subfield of those rational functions $f \in L$ such that $f(x_1, x_2, x_3) = f(x_2, x_3, x_1) = f(x_3, x_1, x_2)$. Prove that L/K is Galois with Galois group cyclic of order 3.

(please continue on other side)

10. Let S be a commutative ring with unit and let R be a subring of S. An element $s \in S$ is said to be *integral* over R if $f(s) = 0$ for some monic polynomial $f(x) \in R[x]$. Prove that the following are equivalent.
    a. $s \in S$ is integral over R.
    b. $R[s]$ is a finitely generated R-module where $R[s]$ is the sub-R-algebra of S generated by s.

11. Let G be a finite group. Prove that $g, h \in G$ are conjugate if and only if $\chi(g) = \chi(h)$ for all irreducible complex characters of G. State clearly any basic results that you are using about characters.

12. Let G be the group of 2×2 invertible upper triangular matrices over a field F. Prove that G is solvable. Is G nilpotent? Justify your answer.

Reminder: please indicate on the cover of your answer book the seven problems you have chosen for grading.