ALGEBRA QUALIFYING EXAM

INSTRUCTIONS: This examination is 3 hours. The exam consists of two parts. Choose a total of **seven problems**, including **at least three from each part**. Indicate on the first page of your exam the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

PART I

1. Let q be a prime power. Let $G = \mathrm{GL}(k, q)$ be the group of invertible matrices with entries in the finite field of order q.
   (a) Show that each element in G with order p is conjugate to an upper triangular matrix in G with 1's along the diagonal.
   (b) When $k = 2$, show that the number of p-Sylow subgroups in G is $p + 1$.
2. (a) For every nonzero power series $f(t) = \sum_{i=0}^{\infty} a_i t^i$, let $v(f)$ be the smallest integer k such that $a_k \neq 0$. Define $v(0) = \infty$. Prove that, for all $f, g \in R[[t]]$,
   $$v(f + g) \geq \min(v(f), v(g))$$
   and $v(fg) = v(f) + v(g)$.
   (b) For every integer $n \geq 2$, compute the inverse of the power series $f(t) = 1 - t^n$ in the power series ring $\mathbb{Z}[[t]]$.
3. (a) Let A be a nonempty subset of a group G. Prove that A is a subgroup if and only if $xA = A$ for all $x \in A$.
   (b) Let G be a group, and let a and b be elements of G such that $a^{-1}ba = b^2$ and $b^{-1}ab = a^2$. Prove that $a = b = e$.
4. A group G is *residually finite* if, for every $g \in G$, $g \neq e$, there exists a finite group G_0 and a homomorphism $\varphi : G \to G_0$ such that $\varphi(g) \neq e$.
   (a) Prove that G is residually finite if and only if, for every $g \in G$, $g \neq e$, there exists a normal subgroup N of G of finite index such that $g \notin N$.
   (b) Let $\mathcal{M}$ be the set of subgroups of G of finite index. Prove that G is residually finite if and only if $\bigcap_{N \in \mathcal{M}} N = \{e\}$.
5. Determine the splitting field for the polynomial $t^3 - 3$ over $\mathbb{Q}$ and compute a primitive element for this field.
6. Compute the Galois groups of the following polynomials over $\mathbb{Q}$:
   (a) $f(t) = t^2 + 2$
   (b) $f(t) = t^3 - 8$

PART II

1. Let R be a commutative ring with identity and let M be a flat R-module. Show that if R is an integral domain then M is torsion-free. Demonstrate by an example that M can have torsion when R is not a domain.

2. Let R, S, T be three integral domains and assume that both S and T are R-algebras. Prove or disprove the following statement: the R-algebra $S \otimes_R T$ (with the natural ring structure) is an integral domain.

3. Compute the rational canonical forms of the following two matrices and based on this decide if they are similar.

$$A = \begin{bmatrix} -1 & 2 & 0 \\ -3 & -2 & 1 \\ -3 & 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -7 & 2 & 0 \\ -18 & 5 & 0 \\ -12 & 4 & -1 \end{bmatrix}.$$

4. Show that $\mathbb{Z}[\sqrt{-14}]$ is not a Euclidean domain.

5. Let $A = \{(f, g) \in \mathbb{F}_p[X] \times \mathbb{F}_p[X] \mid f(0) = g(0)\}$.
   (a) Decide if A is a free $\mathbb{F}_p[X]$-module.
   (b) Show that as a ring A is not integrally closed in its total ring of fractions.

6. Let R be a commutative ring with 1 and let $I \subset R$ be a proper ideal. Let $f : R \twoheadrightarrow R/I$ be the canonical epimorphism. Prove that the map $f^* : \operatorname{Spec} R/I \to \operatorname{Spec} R$ given by $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$ is well-defined, continuous with respect to the Zariski topology, and injective.