CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
SPRING 2021
3 hours

Instructions. The exam consists of two parts. Choose a *total of six problems*, including *three from each part.* Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. Let $G = F(a, b)$ be the free group of rank two and let $[a, b] = a^{-1}b^{-1}ab \in G$. Let $N = \langle\langle [a, b] \rangle\rangle \leq G$ be the normal subgroup of G generated by $[a, b]$. Prove that N consists precisely of all the freely reduced words $w \in F(a, b)$ with exponent sum 0 in each of a and b.
2. For each of the following pairs of groups determine whether or not they are isomorphic; carefully justify why.
   a. $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}, \cdot)$
   b. $(\mathbb{R}, +)$ and $(\mathbb{R}^{\times}, \cdot)$
   c. $(\mathbb{Z} \times \mathbb{Z}, +)$ and $(\mathbb{Q} \times \mathbb{Q}, +)$
   d. $(\mathbb{Q}, +)$ and $(\mathbb{Q} \times \mathbb{Q}, +)$
3. a. Find an element $\omega \in \mathbb{Q}(i, \sqrt{7})$ such that $\mathbb{Q}(\omega) = \mathbb{Q}(i, \sqrt{7})$.
   b. For ω from part a., find the minimal polynomial $m_{\mathbb{Q},\omega}(x)$.
4. Prove that there does not exist a finite simple group of order 280.
5. Exhibit a non-zero non-invertible element $r \in \mathbb{Z}[\sqrt{-5}]$ that admits two distinct factorizations as a product of irreducibles (that are distinct up to permutations and associates) in $\mathbb{Z}[\sqrt{-5}]$. Carefully prove that your example satisfies all the required properties.
6. Let $f(x) = x^4 - x^2 - 1 \in \mathbb{Q}[x]$.
   a. Prove that $f(x)$ is irreducible.
   b. Find a splitting field $K \subseteq \mathbb{C}$ for f.
   c. Prove that the Galois group of f is isomorphic to the dihedral group of order 8.

Part II

7. For each of the following polynomials determine whether it is irreducible in the polynomial ring indicated.
   a. $f(x) = 2x^3 - 7x^2 + 2x + 3$ in $\mathbb{Q}[x]$
   b. $g(x) = x^4 + 7x^3 + 8x + 3$ in $\mathbb{Q}[x]$
   c. $h(x) = x^5 + 2t^2x^4 - t$ in $R[x]$ where $R = \mathbb{Z}[t]$.
8. What is the smallest n such that $GL_n(\mathbb{Q})$ contains an element of order 7? Completely justify your answer by providing a matrix of order 7 for your n and proving that no smaller n will work.
9. Let R be a unital ring.
   a. Define what it means for a left R-module M to be injective.
   b. Suppose that each left ideal I of R has a complement, that is, $R = I \oplus I'$ for some left ideal I'. Prove that every left R-module is injective. Hint: use the Baer criterion.

10. a. Find the Jordan canonical form over the complex numbers of the matrix

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

b. Let

$$A = \begin{bmatrix} 2 & 3 & 12 \\ 4 & 0 & -6 \\ 2 & 3 & 4 \end{bmatrix}$$

and view it as a homomorphism $\mathbb{Z}^3 \to \mathbb{Z}^3$. Express the quotient group $\mathbb{Z}^3/A\mathbb{Z}^3$ as the direct sum of cyclic groups of prime power order (where $A\mathbb{Z}^3$ is the image of A).

11. Let R be a commutative ring with unit. We view all left R-modules as R-R-bimodules in the standard way when taking tensor products. Let M be a left R-module.

a. If I is an ideal of R, recall that

$$IM = \left\{ \sum_{i=1}^{n} a_i m_i \mid a_i \in I, m_i \in M \right\}$$

is a submodule of M. Show that there is a well-defined surjective R-module homomorphism $\eta_I \colon I \otimes_R M \to IM$ with $\eta_I(a \otimes m) = am$ for $a \in I$ and $m \in M$.

b. Show that $\eta_R \colon R \otimes_R M \to M$ is an R-module isomorphism. You may not assume this fact even though it was proved in class and the book.

c. Show that if M is flat, then $\eta_I \colon I \otimes_R M \to IM$ is an R-module isomorphism for every ideal I. Hint: construct a commutative diagram

$$\begin{array}{ccc} I \otimes_R M & \longrightarrow & R \otimes_R M \\ \eta_I \downarrow & & \downarrow \eta_R \\ IM & \longrightarrow & M \end{array}$$

d. Show that if $R = \mathbb{Z}$, $I = 2\mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z}$, then $I \otimes_{\mathbb{Z}} M \cong M$ but $IM = \{0\}$.

12. Let G be a finite group

a. Let V be a nonzero $\mathbb{C}G$-module and $V^2 = V \oplus V$. For each $\lambda \in \mathbb{C}$, define $\varphi_\lambda \colon V^2 \to V$ by $\varphi_\lambda(v, w) = v + \lambda w$.

i. Show that φ_λ is a $\mathbb{C}G$-module homomorphism.

ii. Prove that $\ker \varphi_\lambda = \ker \varphi_{\lambda'}$ if and only if $\lambda = \lambda'$. Explain why V^2 has infinitely many submodules.

b. Let V and W be nonisomorphic simple $\mathbb{C}G$-modules.

i. Show that the only simple submodules of $V \oplus W$ are $V \oplus \{0\}$ and $\{0\} \oplus W$. Hint: if S is a simple submodule apply Schur's lemma to the two projections $S \to V$ and $S \to W$.

ii. Explain why, using semisimplicity of $\mathbb{C}G$, the only submodules of $V \oplus W$ are $\{(0,0)\}$, $V \oplus \{0\}$, $\{0\} \oplus W$ and $V \oplus W$.