Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
Fall 2020

Instructions

The exam has three parts. Only the indicated number of questions will be counted to determine your score. *If you end up doing more, you must specify which problems you would like to be graded.* You have $2\frac{1}{2}$ hours to complete your work.

Notation

- $\mathbb{R}$: Set of all real numbers
- $\mathrm{Re}(z), \mathrm{Im}(z)$: The real and imaginary parts of a complex number z
- $\mathbb{C}$: The complex plane
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathcal{U} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$: the upper half-plane
- By a "region" we mean a non-empty connected open set in $\mathbb{C}$
- By a "conformal map" $\Omega \to \Omega'$ we mean a one-to-one holomorphic map of Ω onto Ω'

Part A. Answer any two of the following three questions.

A1. Give precise statements of the following: (i) Schwarz's lemma (ii) Riemann mapping theorem (iii) The Monodromy theorem (iv) Runge's theorem.

A2. Let $\{u_n\}$ be a sequence of harmonic functions in a region V.

(i) If $u_n \to u$ uniformly compact subsets of V, show that u is harmonic in V.

(ii) If $u_1 \leq u_2 \leq \cdots \leq u_n \cdots$, show that either $\{u_n\}$ converges uniformly on compact subsets of V, or $u_n(z) \to \infty$ for every z in V.

A3. Give a definition of the Green's function for a bounded region Ω.

(i) Find Green's function for the unit disk Δ with singularity at $i/2$.

(ii) Find Green's function for the upper half-plane $\mathcal{U}$ with singularity at $i/2$.

Part B. Solve any one of the following two problems.

B1. Let $f : \mathcal{U} \to \mathcal{U}$ be holomorphic. Prove that
$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \le \frac{1}{\operatorname{Im} z} \text{ for all } z \in \mathcal{U}.$$

B2. Suppose $f_i : \Delta \to \Omega_i$ $(i = 1, 2)$ are two conformal maps where $\Omega_1 \subset \Omega_2$ and $f_1(0) = f_2(0)$. Show that
$$|f_1'(0)| \le |f_2'(0)|$$
and equality holds if and only if $\Omega_1 = \Omega_2$.

PART C. SOLVE ANY FOUR OF THE FOLLOWING SIX PROBLEMS.

C1. Determine the number of zeros of $p(z) = 3z^3 - 2z^2 + 2iz - 8$ in the annulus $1 < |z| < 2$.

C2. Let Ω be a region, and let D be a disc whose closure $\overline{D}$ is contained in Ω. If f is a non constant holomorphic function in Ω such that $|f|$ is constant on the boundary of D, prove that f must have at least one zero in D.

C3. Let g be a meromorphic function on $\mathbb{C}$, with poles of order at most one, and integral residues. Show that there exists a meromorphic function f such that $f'/f = g$.

C4. Let $f : \Delta \to \Delta$ be a surjective holomorphic function with $f(0) = \frac{i}{20}$.

(i) Prove that $f(z)$ has no zeros in the open disk $\{z : |z| < \frac{1}{20.1}\}$.

(ii) Prove that $f(z)$ has at most four zeros in the open disk $\{z : |z| < \frac{1}{2}\}$.

C5. (i) Find a conformal map f from the region $D = \{z : |z| < 1 \text{ and } |z - i| < 1\}$ to the upper half-plane $\mathcal{U}$.

(ii) Is it possible to extend f to a holomorphic function on an open neighborhood of the closure of D. Explain your answer.

C6. Let f be an entire function that satisfies
$$|f(z)| \le a + b|z|^2$$
for all $z \in \mathbb{C}$. Prove that f is a polynomial of degree at most 2.