Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
August 28, 2017

Notations

- $\mathbb{C}$: the complex plane
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathcal{U} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$: the upper half-plane
- $\mathrm{Aut}(\Omega)$: the group of all conformal automorphisms of a domain Ω
- $\mathcal{O}(\Omega)$: the set of all holomorphic functions defined on a domain Ω
- By a "conformal map" f of a region V_1 to a region V_2 we mean a one-to-one holomorphic map f of V_1 onto V_2.
- By a "region" G we mean a nonempty connected open set in $\mathbb{C}$.

PART I: Answer Any TWO Questions.

1. Use Rouche's theorem to prove the fundamental theorem of algebra.

2. (a) Give precise statements of: (i) *Riemann Mapping Theorem*, (ii) *Mittag-Leffler's Theorem*, (iii) *Schwarz's Reflection Principle*.

2. (b) Suppose $f \in \mathcal{O}(\Delta)$, and the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence 1. Prove that f has at least one singular point on the unit circle S^1.

3. State and prove the Poisson's formula for harmonic functions. (You may assume the mean value property for harmonic function.)

4. (a) State the theorem on the existence of entire holomorphic functions with prescribed zeroes.

4. (b) Using (a) prove: Given a region D not equal to $\widehat{\mathbb{C}}$, and a sequence $\{z_n\}$ which does not accumulate in D, prove that there exists $f \in \mathcal{O}(D)$ whose only zeros are $\{z_n\}$.

PART II: Answer Any TWO Questions.

1. Let $f : \Delta \to \Delta$ be holomorphic. Show that

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2} \tag{1}$$

for all $z \in \Delta$. If equality holds in (1) for some $z \in \Delta$, show that $f \in Aut(\Delta)$ and

$$|f'(z)| = \frac{1 - |f(z)|^2}{1 - |z|^2} \tag{2}$$

for all $z \in \Delta$.

2. Let $f : \Delta \to \mathcal{U}$ be holomorphic, with $f(0) = i$. Show that:

$$\text{(i)} \ \frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|} \qquad \text{for all } z \in \Delta;$$

and

$$\text{(ii)} \ |f'(0)| \leq 2.$$

3. Suppose that f is an entire function whose image lies in $\widehat{\mathbb{C}} \setminus \{z : |z| = 1; \text{Im}(z) \geq 0\}$. Prove that f is a constant function.

PART III: Answer Any FOUR Questions.

1. (a) Let V be a simply connected region. Show that a harmonic function in V must have a harmonic conjugate in V.

1. (b) Show that a bounded harmonic function on $\mathbb{C}$ must be a constant.

2. Let f be continuous on a bounded region $\overline{\Omega}$ and holomorphic in Ω. Show that if $\text{Re}(f)$ vanishes on the boundary of Ω then f must be a constant.

3. Suppose f is an entire function and

$$|f(z)| < 1 + |z|^{1/2}$$

for all $z \in \mathbb{C}$. Prove that f is a constant.

4. Let $0 < r < 1$ be fixed and let $D_r = \{z : |z| < r\}$. Prove that there exists $M > 0$ such that for every $z_1, z_2 \in D_r$ and for every positive harmonic function u on the unit disk $\{z : |z| < 1\}$ we have

$$1/M \leq \frac{u(z_1)}{u(z_2)} \leq M.$$

5. Find a conformal map from the region $\{z : |Im(z)| < 1, \ Re(z) > 0\}$ onto $\Delta \setminus (-1, 0]$.

6. Prove that any meromorphic function in the $\mathbb{C}$ is a quotient of two holomorphic functions in $\mathbb{C}$.

7. Assume that $R(z)$ is a rational function which satisfies $|R(z)| = 1$ for all $|z| = 1$. Prove that

$$R(z) = cz^m \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}$$

where c is a constant, m is an integer, and $\alpha_1, \ldots, \alpha_n$ are complex numbers such that $\alpha_k \neq 0$ and $|\alpha_k| \neq 1$.