Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
Fall 2018

Instructions

The exam has three parts. Only the indicated number of questions will be counted to determine your score. *If you end up doing more, you must specify which problems you would like to be graded.* You have 3 hours to complete your work.

Notation

- $\mathrm{Re}(z), \mathrm{Im}(z)$: The real and imaginary parts of a complex number z
- $\mathbb{C}$: The complex plane
- $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: The Riemann sphere
- $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$
- $\mathbb{D}(p, r) = \{z \in \mathbb{C} : |z - p| < r\}$
- $\mathbb{D} = \mathbb{D}(0, 1)$: The unit disk
- $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$
- $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$: The upper half-plane
- $\mathcal{O}(U)$: The space of all holomorphic functions in the open set U.

Part A. Answer any two of the following three questions.

A1. State the classical version of the Schwarz lemma on holomorphic functions $\mathbb{D} \to \mathbb{D}$ which fix the origin. What form would the lemma take if the origin is not assumed fixed?

A2. Let u be a real-valued function which is continuous on $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$. State the Poisson integral formula for u and use it to write a formula for the unique $f \in \mathcal{O}(\mathbb{D})$ which satisfies $\mathrm{Re}(f) = u$ and $f(0) = u(0)$.

A3. State (without proof) at least seven equivalent conditions for a connected, open set in $\mathbb{C}$ to be simply connected.

Part B. Solve any two of the following four problems.

B1. Suppose $f \in \mathcal{O}(\mathbb{C})$ is represented by the power series $\sum_{n=0}^{\infty} a_n z^n$. Under what condition on $\{a_n\}$ does f map the real axis to the imaginary axis? Under what condition on $\{a_n\}$ is f injective?

B2. Give an explicit formula for a conformal map between the slit disk $\mathbb{D} \smallsetminus [0, 1)$ and the half-strip $\{z \in \mathbb{C} : |\,\mathrm{Im}(z)| < 1 \text{ and } \mathrm{Re}(z) > 0\}$.

B3. Let f and g be holomorphic in a neighborhood of the closed disk $\overline{\mathbb{D}(0,R)}$, and assume that f is never zero on the circle $|z| = R$. Prove that there exists $\varepsilon > 0$ such that f and $f + \varepsilon g$ have the same number of zeros in $\mathbb{D}(0,R)$.

B4. Let f be holomorphic and injective in $\mathbb{D}^*$. Show that 0 cannot be an essential singularity of f. Use this fact to prove that $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b : a, b \in \mathbb{C}, a \neq 0\}$.

<u>Part C. Solve any four of the following six problems.</u>

C1. Suppose $f \in \mathcal{O}(\mathbb{H})$ is bounded and maps the horizontal line $\mathrm{Im}(z) = 1$ to the real axis $\mathrm{Im}(z) = 0$. Show that f is constant.

C2. Let h be a positive harmonic function in $\mathbb{D}$. Show that there are harmonic functions $u, v : \mathbb{D} \to \mathbb{R}$ such that $h = e^u \sin v$.

C3. Let $M > 1$ and $\mathcal{F}$ be the family of all $f \in \mathcal{O}(\mathbb{D})$ such that $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq M$ for every $z \in \mathbb{D}$. Show that there exists an $R > 0$ such that $f(\mathbb{D}) \supset \mathbb{D}(0,R)$ for all $f \in \mathcal{F}$.

C4. Let $a_n = 1 - 1/n^2$. Prove that the infinite product

$$f(z) = \prod_{n=1}^{\infty} \left(\frac{a_n - z}{1 - a_n z} \right)$$

defines a holomorphic function $f : \mathbb{D} \to \mathbb{D}$.

C5. Show that for every $q \in \mathbb{C}$ the equation $\sin z = qz$ has infinitely many solutions.

C6. If $f : \mathbb{D} \to \mathbb{D}^*$ is holomorphic, prove that $|f'(0)| \leq 2/e$.