Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
Spring 2013

Notations

- $\mathrm{Re}(z)$: the real part of a complex number z
- $\mathbb{C}$: the complex plane
- $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: the Riemann sphere
- $\Delta = \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathrm{Aut}(\Omega)$: the group of all conformal automorphisms of Ω
- $\mathcal{O}(V)$: the set of all holomorphic functions on V
- $B(0,R) := \{z \in \mathbb{C} : |z| < R\}$
- $\overline{B(0,R)} := \{z \in \mathbb{C} : |z| \leq R\}$

PART I: Answer Any TWO Questions

1. (i) Let $\Omega \neq \mathbb{C}$ be a simply connected region, and let $a \in \Omega$. Suppose there are holomorphic functions $f : \Omega \to \Delta$ and $g : \Omega \to \Delta$ such that : (i) $f(a) = g(a) = 0$, (ii) $f'(a) > 0, g'(a) > 0$, and (iii) both f and g are one-to-one and onto. Show that $f = g$.

(ii) Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $\Delta^* := \Delta \setminus \{0\}$. State Aut $\mathbb{C}^*$ and Aut Δ^* (no proofs required).

2. (i) Give precise statements of the following: (a) the *Mittag-Leffler theorem* (on the existence of meromorphic functions in the plane with prescribed principal parts) (b) the *Weierstrass theorem* (on the zeroes of holomorphic functions in an open set G in $\widehat{\mathbb{C}}$, $G \neq \widehat{\mathbb{C}}$).

(ii) Let $f \in \mathrm{Aut}(\Delta)$. Show that $f(z) = e^{i\theta}\varphi_\alpha(z)$ for all $z \in \Delta$, where $0 \leq \theta < 2\pi$, $f^{-1}(\alpha) = 0$ and

$$\varphi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z} \qquad \text{for all } z \in \Delta.$$

3. (i) Give a precise definition of *natural boundary*. If Ω is a region, show that there always exists an $f \in \mathcal{O}(\Omega)$ which has no holomorphic extension to any larger region.

(ii) State the *monodromy theorem*.

4. Let $\{u_n\}$ be a sequence of harmonic functions in a region V. Prove the following:

(i) If $u_n \to u$ uniformly on compact subsets of V, then u is harmonic in V.

(ii) If $u_1 \leq u_2 \leq u_3 \leq \cdots$, then either $\{u_n\}$ converges uniformly on compact subsets of V, or $u_n(z) \to \infty$ for every $z \in V$.

PART II: Answer Any TWO Questions

1. Let $f \in \mathrm{Aut}(\Delta)$ such that f has two distinct fixed points in Δ. What can you say about f? Prove your claim.

2. Show that any holomorphic function $f : \Delta \to \overline{\Delta}$ must satisfy

$$|f(0)| + \frac{(1-|z|)}{2|z|}\left|f(z) - f(0)\right| \leq 1 \qquad \text{for all } z \in \Delta \setminus \{0\}.$$

3. For any z, a in Δ we define

$$d(z,a) = \left|\frac{z-a}{1-\bar{a}z}\right|.$$

Show that d is a metric on Δ.

PART III: Answer Any FOUR Questions

1. Let V be a region and let A be a discrete and (relatively) closed subset of V. Let $f : V \setminus A \to \mathbb{C}$ be holomorphic and injective. Show that:

(i) no point $c \in A$ can be an essential singularity of f.

(ii) if $c \in A$ is a pole of f, then c must have order 1.

2. Show that for every real number $\lambda > 1$, the holomorphic function $f(z) = ze^{\lambda - z} - 1$ has exactly one zero in Δ, and that it is real and positive.

3. Let V be a region, and D be a disk, such that $\overline{D} \subset V$. Suppose $f \in \mathcal{O}(V)$, f is not constant, and $|f|$ is constant on the boundary ∂D. Does there exist some $z \in D$, such that $f(z) = 0$? Prove your claim.

4. Let S denote the class of all functions $f \in \mathcal{O}(\Delta)$ which are univalent in Δ and satisfy $f(0) = 0$ and $f'(0) = 1$. If $f \in S$, show that there exists a $g \in S$ such that $g^2(z) = f(z^2)$ for all $z \in \Delta$.

5. Suppose Ω is a simply connected region, $f \in \mathcal{O}(\Omega)$, f has no zero in Ω, and n is a positive integer. Prove that there exists a $g \in \mathcal{O}(\Omega)$ such that $g^n = f$.

6. (i) Define an *elliptic function*.

(ii) Show that a non-constant elliptic function cannot be entire.

(iii) Show that the number of poles of a non-constant elliptic function counted with multiplicity is greater than 1.

7. Show that

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

for all $z \in \mathbb{C}$ and that the convergence is uniform on compact subsets of $\mathbb{C}$.

8. (i) Find a conformal mapping from the strip

$$U = \{x + iy : -\infty < x < \infty, 0 < y < \pi\}$$

onto the half-strip

$$V = \{x + iy : 0 < y < \pi/2, 0 < x < \infty\}.$$

(ii) Let a be a positive real number. Let U be the open set obtained from the complex plane by deleting the segment $a \le x < \infty$. Find a conformal mapping of U onto Δ.