Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam

Spring 2018

<u>Instructions</u>

The exam has three parts. Only the indicated number of questions will be counted to determine your score. *If you end up doing more, you must specify which problems you would like to be graded.* You have 3 hours to complete your work.

<u>Notation</u>

- $\mathrm{Re}(z), \mathrm{Im}(z)$: The real and imaginary parts of a complex number z
- $\mathbb{C}$: The complex plane
- $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: The Riemann sphere
- $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$
- $\mathbb{D}(p, r) = \{z \in \mathbb{C} : |z - p| < r\}$
- $\mathbb{D} = \mathbb{D}(0, 1)$: The unit disk
- $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$
- $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$: The upper half-plane
- $\mathcal{O}(U)$: The space of all holomorphic functions in the open set U.

<u>Part A. Answer any two of the following three questions.</u>

A1. Describe, without proof, the group of conformal automorphisms of each of the following domains: $\widehat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{H}$, $\mathbb{D}$, $\mathbb{D}^*$.

A2. Define the class $\mathcal{S}$ of *schlicht* functions in the unit disk. What does Koebe's 1/4-theorem assert about schlicht functions? Give an example of a schlicht function which shows that the constant 1/4 in this theorem is optimal.

A3. What does it mean for a function $f \in \mathcal{O}(\mathbb{D})$ to have the unit circle $\partial\mathbb{D}$ as its natural boundary? Outline two methods of producing such an f.

<u>Part B. Solve any two of the following four problems.</u>

B1. Let $U_\varepsilon = \mathbb{D}(1, \varepsilon)$. Show that for every $\varepsilon > 0$ there is a conformal map $f : \mathbb{D} \to \mathbb{D}$ such that $f(\mathbb{D} \smallsetminus U_\varepsilon) \subset U_\varepsilon$.

B2. Determine the radius of convergence of the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \text{where } a_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

Then find a formula for $f(z)$ in terms of familiar elementary functions.

B3. Compute the following integral by the method of residues:

$$\int_0^{\infty} \frac{x \sin x}{x^2+1}\, dx.$$

B4. Let $0 < \alpha < 2\pi$. Find an explicit formula for a conformal map from $S = \{z \in \mathbb{C} : \operatorname{Re}(z) < 0,\ 0 < \operatorname{Im}(z) < \pi\}$ onto $W = \{z \in \mathbb{C} : |z| < 1,\ 0 < \arg(z) < \alpha\}$.

PART C. SOLVE ANY FOUR OF THE FOLLOWING SIX PROBLEMS.

C1. Characterize all $f \in \mathcal{O}(\mathbb{C})$ for which $|f(z)| = 1$ whenever $|z| = 1$.

C2. If f is holomorphic in a neighborhood of the closed unit disk $\overline{\mathbb{D}}$ and $u = \operatorname{Re}(f)$, show that

$$f(z) = \frac{1}{\pi i} \int_{|\zeta|=1} \frac{u(\zeta)}{\zeta - z}\, d\zeta - \overline{f(0)}$$

for all $z \in \mathbb{D}$.

C3. Suppose $f_n : \mathbb{D} \to \mathbb{D}$ is a sequence of holomorphic functions such that $f_n(0) \to 1$. Show that $f_n \to 1$ uniformly on compact subsets of $\mathbb{D}$.

C4. Suppose $U \subset \mathbb{C}$ is a bounded domain and $p \in U$. Let $\mathcal{F}$ be the family of all injective holomorphic functions $f : \mathbb{D} \to U$ with $f(0) = p$. Prove the following assertions:

(i) There is a $g \in \mathcal{F}$ with the property $|g'(0)| = \sup_{f \in \mathcal{F}} |f'(0)|$.

(ii) If V is any simply connected domain with $g(\mathbb{D}) \subset V \subset U$, then $g(\mathbb{D}) = V$.

C5. Show that for every $q \in \mathbb{C}$ the equation $e^{2z} + e^{3z} = q$ has infinitely many solutions.

C6. Suppose $f \in \mathcal{O}(\mathbb{D})$ satisfies $f(0) = 0$ and $|f(z) + zf'(z)| < 1$ for all $z \in \mathbb{D}$. Show that $|f(z)| \le |z|/2$ for all $z \in \mathbb{D}$.