Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam

Spring 2019

Instructions

The exam has three parts. Only the indicated number of questions will be counted to determine your score. *If you end up doing more, you must specify which problems you would like to be graded.* You have 3 hours to complete your work.

Notation

- $\mathrm{Re}(z), \mathrm{Im}(z)$: The real and imaginary parts of a complex number z
- $\mathbb{C}$: The complex plane
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathcal{U} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$: the upper half-plane
- $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$: the right half-plane
- By a "region" we mean a non-empty connected open set in $\mathbb{C}$
- By a "conformal map" $\Omega \to \Omega'$ we mean a one-to-one holomorphic map of Ω onto Ω'

PART A. ANSWER ANY TWO OF THE FOLLOWING THREE QUESTIONS.

A1. Describe, without proof, the group of conformal automorphisms of the following regions: $\mathbb{C}$, $\mathbb{C} \smallsetminus \{0\}$, Δ, $\Delta \smallsetminus \{0\}$, $\mathcal{U}$.

A2. Prove that every meromorphic function on the Riemann sphere is a rational function.

A3. If G is a bounded Dirichlet region, show that for each $a \in G$ there is a Green's function on G with its singularity at a.

PART B. SOLVE ANY TWO OF THE FOLLOWING FOUR PROBLEMS.

B1. Let f and g be entire functions that satisfy $|f(z)| \geq 2|g(z)|$ for all $z \in \mathbb{C}$. Show that there exists $c \in \mathbb{C}$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

B2. Let $f : \Delta \to \mathcal{H}$ be holomorphic, such that $f(0) > 0$. Show that

$$f(0)\,\frac{1-|z|}{1+|z|} \leq |f(z)| \leq f(0)\,\frac{1+|z|}{1-|z|}$$

for all $z \in \Delta$.

B3. Suppose $f : \Delta \to \Delta \smallsetminus \{0\}$ is holomorphic. Prove that

$$|f(z)| \leq |f(0)|^{\frac{1-|z|}{1+|z|}} \qquad \text{for all } z \in \Delta.$$

B4. Compute

$$\frac{1}{2\pi i} \int_\gamma \frac{z}{z^2 - 1}\, dz$$

where γ is a smooth closed curve in $\mathbb{C}$ that avoids ± 1.

Part C. Solve any four of the following six problems.

C1. Let f be an entire function which is periodic in the sense that $f(z + \omega) = f(z)$ for some $\omega \neq 0$. Show that f has a fixed point.

C2. Let $\Omega \subset \mathbb{C}$ be a bounded region (not necessarily simply connected) containing the origin. Suppose $f : \Omega \to \Omega$ is holomorphic with $f(0) = 0$ and $f'(0) = 1$. Show that $f(z) = z$ for all $z \in \Omega$.

C3. Prove that

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

using the canonical representation.

C4. Let u be a real-valued harmonic function in a region Ω. Prove that if u^2 is also harmonic in Ω, then u is constant.

C5. Prove that, if $1 < a < \infty$, the function $z + a - e^z$ has only one zero in the left half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$, and that this zero is on the real axis.

C6. Show that there exists a sequence of polynomials $p_n(z)$ such that $p_n(z) \to 1$ if $\mathrm{Im}(z) > 0$, $p_n(z) \to 0$ if $\mathrm{Im}(z) = 0$, and $p_n(z) \to -1$ if $\mathrm{Im}(z) < 0$.