Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
Spring 2021

Instructions

The exam has three parts. Only the indicated number of questions will be counted to determine your score. *If you end up doing more, you must specify which problems you would like to be graded.* You have 2 1/2 hours to complete your work.

Notation

- $\mathbb{R}$: Set of all real numbers
- $\mathrm{Re}(z), \mathrm{Im}(z)$: The real and imaginary parts of a complex number z
- $\mathbb{C}$: The complex plane
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathcal{U} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$: the upper half-plane
- By a "region" we mean a non-empty connected open set in $\mathbb{C}$
- By a "conformal map" $\Omega \to \Omega'$ we mean a one-to-one holomorphic map of Ω onto Ω'

PART A. ANSWER ANY TWO OF THE FOLLOWING THREE QUESTIONS.

A1. (i) Give precise statement of the Riemann mapping theorem.
(ii) Give precise statement of the Schwarz-Christoffel formula.
(iii) Indicate at least three main steps and how they give a proof of Riemann mapping theorem.

A2. Give precise statements of the following: (i) The Mean Value property of harmonic functions (ii) The Harnack's inequality for harmonic functions.

A3. Prove the fundamental theorem of algebra.

PART B. SOLVE ANY ONE OF THE FOLLOWING TWO PROBLEMS.

B1. Let $f : \mathcal{U} \to \mathcal{U}$ be holomorphic such that $f(i) = i$. Give the lowest possible upper bound to $|f(2i)|$ and $|f'(i)|$. (Hint: Use Schwarz Lemma.)

B2. Let $f(z)$ be an entire function that satisfies

$$|f(z)| \le \log(1 + |z|)^{10}.$$

Prove that $f(z) \equiv 0$.

C1. Find a conformal map from $\mathbb{C} \setminus \{z = x + iy : |z| = 1, \ y \geq 0\}$ onto $\{z = x + iy : |z| > 1\}$.

C2. Let f be meromorphic on $\mathbb{C}$, and suppose that
$$\lim_{|z| \to \infty} |f(z)| = \infty.$$
Prove that f is a rational function.

C3. Let a be a real number > 1. Prove that the equation
$$ze^{a-z} = 1$$
has a single solution in Δ, which is real and positive.

C4. Let Ω be a region in $\mathbb{C}$ and let $\mathcal{F}$ be a family of all homorphic functions $f : \Omega \to \mathbb{C}$ such that
$$\iint_\Omega |f(z)| dx dy < \infty.$$
Prove that $\mathcal{F}$ is a normal family.
   If the above condition is replaced by $\iint_\Omega |f(z)|^2 dx dy < \infty$ does the family remain normal?

C5. Let f be an entire function. If $z^{-3} \ \mathrm{Re}[f(z)] \to 0$ as $z \to \infty$, prove that $f(z)$ is a polynomial of degree 2.

C6. Prove that an elliptic function cannot have a single simple pole in its fundamental polygon.