Qualifying Exam in Complex Analysis, May 2001

May 18, 2001

Instructions: Do eight problems. State clearly any results you use.

1. (a) Give an example of two simply connected domains Ω and Ω' in $\mathbb{C}$ which are real C^∞ diffeomorphic, but are not biholomorphically equivalent. Explain.

   (b) Construct a biholomorphic equivalence between the unit disk, D and the upper half plane, P.

2. Let u be harmonic on a connected domain Ω in $\mathbb{C}$

   (a) Show $f = u_x - iu_y$ is holomorphic in Ω.

   (b) Suppose u is the real part of a holomorphic function g in Ω. Show $g' = f$.

   (c) Give a necessary and sufficient condition on Ω so that every harmonic function is the real part of a holomorphic function. Justify your answer.

3. Calculate $\int_{-\infty}^{\infty} \frac{x^2dx}{(1+x^2)^2}$ using the residue theorem. Justify all the steps in your calculation.

4. (a) Let Ω be a connected domain in $\mathbb{C} = \mathbb{R}^2$ and f a smooth local diffeomorphism $f : \Omega \to \mathbb{R}^2$. What does it mean to say f is conformal?. Give the geometric definition.

   (b) Show that $f : \Omega \to \mathbb{C}$ is conformal if and only f is holomorphic and f' never vanishes on Ω.

5. (a) Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on a domain, Ω. Let $\Omega^- = \{z \in \mathbb{C} : z^- \in \Omega\}$. Define $f^* : \Omega \to \mathbb{C}$ by $f^*(z) = f(z^-)^-$. Show f^* is holomorphic on Ω^-.

   (b) Suppose Ω is connected and non-empty and $\Omega^- = \Omega$. Show $\Omega \cap \mathbb{R}$ is non-empty, and, if f is holomorphic in Ω, then $f = f^*$ if and only if $f(x)$ is real for all $x \in \Omega \cap \mathbb{R}$.

6. Let f be a non-constant holomorphic function on Ω, where Ω is a bounded connected domain and $|f|$ is constant on $\partial(\Omega)$. Show f must have a zero in Ω. (assume f is cont. on $\bar{\Omega}$)

7. (a) State Schwartz' lemma.

   (b) Let $Aut(D)$ be the group of holomorphic automorphisms of the unit disk D. Find $Aut(D)$

8. Let u be a harmonic function on a connected domain Ω containing a disk, D which is identically zero on the boundary ∂D. Show u is identically zero on D. Show if two harmonic functions agree on ∂D, then they must coincide on Ω.

9. Let $D(a, r_0)$ denote the open disk $\{z : |z - a| < r_0\}$.

   (a) Suppose f is holomorphic in $D(a, r_0)$. Show that (*) there exists a non-decreasing function $M : (0, r_0) \to (0, \infty)$ such that $|f^{(n)}(a)| \leq \frac{n!M(r)}{r^n}$ for all integers $n \geq 0$ and $r \in (0, r_0)$.

   (b) Suppose f is holomorphic in $D(a, r_1)$ for some $r_1 \in (0, r_0)$ and satisfies (*) above. Show f extends holomorphically to $D(a, r_0)$.

10. Let $D = \{z : |z| < 1\}$. Suppose f_n is a sequence of holomorphic functions on D and $\exp(f_n(z)) \to g(z)$ uniformly on compacta of D. If $g(0) = 0$, what can be said about g? Explain.

11. For a domain Ω in $\mathbb{C}$, a subset A of Ω is said to be *locally finite* in Ω if $A \cap K$ is finite for each compact set K in Ω. Suppose A is locally finite in Ω and f_n is a sequence of holomorphic functions on Ω which converges uniformly on compacta on $\Omega - A$.

    (a) Show $\Omega - A$ is open.

    (b) Show there is a unique holomorphic function f on Ω such that f_n converges uniformly on compacta to f on Ω.

12. Suppose Ω is a simply connected domain and g is a holomorphic function in it which is not identically zero. Let $n > 1$ be an integer. Show g has a holomorphic n^{th} root on Ω if and only if every zero of g in Ω has multiplicity divisable by n.

13. Suppose f is holomorphic on an open set containing D^-, where D is the open disk $\{z : |z| < 1\}$ and that f has at least 2 zeros (or one zero with multiplicity at least 2) in $\{z : |z| \leq \frac{1}{2}\}$. Show $|f(0)| \leq \frac{1}{4}$. (assume $|f(z)| \leq 1$ for $|z| \leq 1$)

14. For $k = 0, 1, \ldots$ show that $\int_0^\infty t^k e^{-zt}\,dt$ converges in the right half plane, $R = \{z : \Re z > 0\}$ to a holomorphic function $F_k(z)$ and in fact $F_k(z) = \frac{k!}{z^{k+1}}$ for z$\in R$. Hint: Show $F_{k+1}(z) = -F_k'(z)$ for all $k = 0, 1, \ldots$ and $z \in R$.