Differential Geometry
Exercises

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Jordan curve theorem

We think of a regular C^2 simply closed path in the plane as a C^2 imbedding of the circle $\omega : \mathbb{S}^1 \to \mathbb{R}^2$.

Theorem. *Given the C^2 imbedding ω of $\mathbb{S}^1$ into $\mathbb{R}^2$, the complement of the image $\mathbb{R}^2 \setminus \omega(\mathbb{S}^1)$ has precisely two components, one of which is bounded and one unbounded.*

Exercise 1. Prove that $\mathbb{R}^2$ has at most two components:

(a) Let $\mathbb{S}^1$ have its usual orientation, let $\mathbf{v}$ be the unit velocity vector field along ω given by the orientation, let ι denote the rotation of $\mathbb{R}^2$ by $\pi/2$ radians, and let $\mathbf{n}$ be the unit vector field along ω given by

$$\mathbf{n} = \iota\mathbf{v},$$

that is, $\mathbf{n}$ is the unit normal vector field along ω consistent with the imbedding of ω in $\mathbb{R}^2$. Define

$$F : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}^2$$

by

$$F(t, \epsilon) = \omega(t) + \epsilon\mathbf{n}(t),$$

where $(t, \epsilon) \in \mathbb{S}^1 \times \mathbb{R}$. Show that F has maximal rank on $\mathbb{S}^1 \times \{0\}$.

(b) Show that there exists $\epsilon > 0$ such that $F|\mathbb{S}^1 \times (-\epsilon_o, \epsilon_o)$ is a diffeomorphism of $\mathbb{S}^1 \times (-\epsilon_o, \epsilon_o)$ onto its image, an open neighborhood of $\omega(\mathbb{S}^1)$ in $\mathbb{R}^2$.

(c) Conclude that $\mathbb{R}^2 \setminus \omega(\mathbb{S}^1)$ has at most two components.

Exercise 2. Prove that $\mathbb{R}^2 \setminus \omega(\mathbb{S}^1)$ has at least two components: the method is to construct a function on $\mathbb{R}^2 \setminus \omega(\mathbb{S}^1)$ such that the function is constant on each components of $\mathbb{R}^2 \setminus \omega(\mathbb{S}^1)$, and that the function assumes at least two values.

(a) Introduce polar coordinates into $\mathbb{R}^2$ centered at any point $p = (a, b)$ in $\mathbb{R}^2$.

(b) Show that if there exists a path $\gamma : (\alpha, \beta) \to \mathbb{R}^2 \setminus \{p\} \in C^2$, then there exist C^2 functions $r : (\alpha, \beta) \to (0, \infty)$ and $\sigma : (\alpha, \beta) \to \mathbb{R}$ such that

$$\gamma(\tau) = p + r(\tau)e^{i\sigma(\tau)}.$$

Check as to what extent the function $\sigma(\tau)$ is unique. Show, that when $\sigma \in C^1$, the function $\sigma'(\tau)$ is uniquely defined, and is independent of the particular choice of $\sigma(\tau)$.

(c) Parametrize $\mathbb{S}^1$ by

$$t = e^{i\tau}, \tau \in \mathbb{R},$$

and let

$$\gamma(\tau) = \omega(e^{i\tau}).$$

For any $p \in \mathbb{R}^2 \setminus \omega(\mathbb{S}^1)$, let $\sigma_p(\tau)$ be the angle function as defined in (b) above, and define the *winding number* $\mathfrak{n}_\omega(p)$ *of* ω *with respect to* p, by

$$\mathfrak{n}_\omega(p) = \frac{\sigma_p(2\pi) - \sigma_p(0)}{2\pi}.$$

Show that (i) $\mathfrak{n}_\omega(p)$ is an integer; (ii) $\mathfrak{n}_\omega(p)$ is independent of the particular choice of the angle function σ_p; (iii) $\mathfrak{n}_\omega(p)$ is a C^k function of p on $\mathbb{R}^2 \setminus \omega(\mathbb{S}^1)$; and (iv) if $k \geq 1$, then $\mathfrak{n}_\omega$ changes its values by $+1$ or -1 when p crosses ω at any of its image points in $\mathbb{R}^2$.

(d) Finish the proof of the Jordan curve theorem.

Whitney's theorem

Exercise 3. Let $n = 2$; so we are discussing paths and curves in the plane. Let ι denote the rotation of $\mathbb{R}^2$ by $\pi/2$ radians. Then we may define the curvature with sign, namely,

$$\kappa = \frac{\mathbf{N}\cdot\iota(\omega'/|\omega'|)}{|\omega'|^2} = \frac{\omega''\cdot\iota\omega'}{|\omega'|^3}.$$

(a) Show that κ, here, is defined along the oriented curve determined by ω.

(b) Let s denote arc length along ω, that is, reparametrize ω with respect to arc length. Let $\mathbf{t}$ denote the unit velocity vector field along ω, that is,

$$\mathbf{t} = \frac{\omega'}{|\omega'|},$$

and $\mathbf{n}$ the positively oriented unit normal vector field along ω, that is,

$$\mathbf{n} = \iota\mathbf{t}.$$

Prove the *Frenet formula*:

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t}.$$

Exercise 4. Write the Frenet formulae as a matrix ordinary differential equation, namely,

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}.$$

(a) Show that if we start with s the arc length and $\kappa(s)$ the curvature function then there exists a function $\theta(s)$ such that

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}(s) = \begin{pmatrix} \cos\theta(s) & \sin\theta(s) \\ -\sin\theta(s) & \cos\theta(s) \end{pmatrix}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

(b) Determine to what extent is the function $\theta(s)$ is uniquely determined by the curvature function $\kappa(s)$.

(c) Since s is arc length, $\mathbf{t}(s)$ is the velocity vector field of the parametrization of the oriented curve, determined by ω, with respect to arc length. Therefore determine the full extent to which an oriented curve is uniquely determined by its curvature as a function of arc length along the curve.

Definition. A path $\omega : \mathbb{R} \to \mathbb{R}^2$ is *closed* if ω is periodic. Given the path $\omega : \mathbb{R} \to \mathbb{R}^2$, we say that ω is *simply closed* if there exists a real number $\ell \neq 0$ such that

$$\omega(t_1) = \omega(t_2) \quad \Longleftrightarrow \quad \frac{t_2 - t_1}{\ell} \in \mathbb{Z},$$

where $\mathbb{Z}$ denotes the integers.

Given any closed path $\omega : \mathbb{R} \to \mathbb{R}^2$ of period $\ell > 0$, we define its *index of rotation* rot_ω by

$$\mathrm{rot}_\omega = \frac{1}{2\pi} \int_0^\ell \kappa(t)|\omega'(t)|\,dt = \frac{1}{2\pi} \int_0^{s_o} \kappa\,ds,$$

where

$$s_o = \int_0^\ell |\omega'(t)|\,dt$$

is the length of the closed curve.

Exercise 5. Show that rot_ω is an integer.

Definition. let $\omega : \mathbb{R} \to \mathbb{R}^2$ and $\gamma : \mathbb{R} \to \mathbb{R}^2$ be two regular ℓ–periodic C^2 paths in the plane. We say that

$$\Omega : \mathbb{R} \times [0, 1] \to \mathbb{R}^2 \in C^2$$

is a deformation of ω *to* γ if

$$\Omega|\mathbb{R} \times \{0\} = \omega, \qquad \Omega|\mathbb{R} \times \{1\} = \gamma,$$

and if for each $t \in [0, 1]$, we have $\Omega|\mathbb{R} \times \{t\}$ is a regular ℓ–periodic C^2 path in the plane.

Exercise 6.

(a) Show that "deformation of ω into γ" is an equivalence relation.

(b) Show that the rotation number of a regular ℓ–periodic C^2 path in the plane is actually defined on the equivalence class (relative to the "deformation" relation) of the path.

Exercise 7. Show that, for any given $\rho > 0$, integer $k \neq 0$, the ℓ–periodic path

$$\gamma_{k;\rho,\ell}(t) = \frac{\rho}{2\pi k} \left\{ \cos \frac{2\pi kt}{\ell}\mathbf{i} + \sin \frac{2\pi kt}{\ell}\mathbf{j} \right\}$$

has length ρ, and rotation index k.

Exercise 8. Prove

Theorem: *Any two regular ℓ–periodic C^2 paths in the plane, possessing the same nonzero rotation number, may be deformed one into the other.*

Here is a breakdown of the steps:

(a) Characterize the velocity vector field $\mathbf{v}(t)$ of an ℓ–periodic closed curve by

$$\int_0^\ell \mathbf{v}(t)\,dt = 0.$$

(b) Given any positive constant α, show that any regular ℓ–periodic C^2 path may be deformed to an ℓ–periodic path whose velocity vector has any *constant* length α.

(c) Therefore, assume the two paths ω_1, ω_2 have nonzero rotation index k, period ℓ, and velocity vector of constant length 1; and let κ_1, κ_2 denote the curvature functions of ω_1, ω_2, respectively. Deform $\kappa_1(t)$ to $\kappa_2(t)$ by

$$\kappa_\epsilon(t) = (1-\epsilon)\kappa_1(t) + \epsilon\kappa_2(t), \qquad \epsilon \in [0,1].$$

(d) Integrate the deformation of curvature functions (in (c)), using Exercise 4(c), to construct the deformations of the paths from the deformation of the curvature functions.

Surfaces in Euclidean space

Exercise 9. Show that every compact surface in $\mathbb{R}^3$ has a point of strictly positive curvature.

Exercise 10. Show that there exists a sequence of compact surfaces $\mathcal{S}_n$ in $\mathbb{R}^3$, with induced Riemannian metric such that

$$A(\mathcal{S}_n) \to 0 \ \text{ as } \ n \to +\infty,$$

but the collection of surfaces is not contained in any compact set.

Riemannian volume

Exercise 11. Let M be a Riemannian manifold, ξ a C^1 vector field on M, $\Phi_t : M \to M$ the 1–parameter flow determined by ξ. Prove

$$\left.\frac{d}{dt}\right|_{t=0} V(\Phi_t(D)) = \iint_D \operatorname{div}\xi \, dV,$$

for any relatively compact D in M.

Exercise 12. Let M^n be a Riemannian manifold, $x : U \to \mathbb{R}^n$ a chart on M, $\mathcal{G} = (g_{ij})$ the positive definite symmetric matrix representing the Riemannian metric in the chart.

(a) Show that

$$dV = \sqrt{\det \mathcal{G}}\, dx^1 \cdots dx^n$$

is independent of the chart x.

(b) Let $\{e_1, \ldots, e_n\}$ denote an orthonormal moving frame on U, with dual co-frame $\{\omega^1, \ldots, \omega^n\}$, and skew-symmetric matrix of connection 1–forms $({\omega_j}^k)$. Show that

$$\omega^1 \wedge \cdots \omega^n = \pm\sqrt{\det \mathcal{G}}\, dx^1 \wedge \cdots \wedge dx^n,$$

the $\pm$ depending on whether the frame has the same, or opposite, orientation as the chart.

Exercise 13. Let ω be the $(n-1)$–form on $\mathbb{R}^n$ given by

$$\omega = \sum_{j=1}^{n} (-1)^{j-1} x^j \, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n.$$

(a) Show that ω is invariant under the orthogonal group of $\mathbb{R}^n$.

(b) Show that $\omega|\mathbb{S}^{n-1}(r)$ is $\pm$ the $(n-1)$–dimensional volume form on $\mathbb{S}^{n-1}(r)$ for every $r > 0$.

The Levi–Civita connection

Exercise 14. Continue with the previous exercise. Let $\{E_1, \dots, E_n\}$ denote another orthonormal moving frame on U, with dual co-frame $\{\theta^1, \dots, \theta^n\}$, and skew-symmetric matrix of connection 1–forms $({\theta_j}^k)$. Give a formula relating the connection matrices $({\omega_j}^k)$ and $({\theta_j}^k)$.

Exercise 15. Let M be a Riemannian manifold. First show that, given any point $p \in M$, there exists a chart about p such that the Christoffel symbols all vanish at p, that is,

$$\Gamma_{jk}{}^{\ell}(p) = 0, \qquad \text{for all } \ j, k, \ell.$$

Then show that one can also assume that

$$g_{jk}(p) = \delta_{jk} \quad \text{(the Kronecker delta)},$$

that is, the natural basis at p of M_p is orthonormal.

HINT: Given a chart $x : U \to \mathbb{R}^n$, $p \in U$, with Christoffel symbols $\Gamma_{jk}{}^{\ell}$ symmetric in j, k, define

$$y^\ell(q) = x^\ell(q) - x^\ell(p) + \frac{1}{2} \sum_{j,k} \Gamma_{jk}{}^{\ell}(x(p))(x^j(q) - x^j(p))(x^k(q) - x^k(p)).$$

Check that there exists a domain V in U, such that $(y|V) : V \to \mathbb{R}^n$ is a chart for which the Christoffel symbols vanish at p.

Riemannian submanifolds

Exercise 16.

(a) Let M be a submanifold of $\overline{M}$. We say that M is *totally geodesic in* $\overline{M}$, if for any geodesic γ in $\overline{M}$, for which there exists t_0 such that $\gamma(t_0) \in M$ and $\gamma'(t_0) \in M_{\gamma(t_0)}$, there exists an $\epsilon > 0$ such that $\gamma|(t_0 - \epsilon, t_0 + \epsilon)$ is completely contained in M. Show that M is totally geodesic if and only if the second fundamental form B vanishes identically on M.

(b) Show that if M is a Riemannian manifold possessing an isometry $\phi : M \to M$, then any connected component of the set of all points left fixed by ϕ is totally geodesic.

Exercise 17. Let M be a codimension 1 submanifold of the Riemannian manifold $\overline{M}$, $p \in M$. Let ξ be a unit vector orthogonal to M_p. Let $\tilde{M}_p = \exp M_p$, where exp denotes (for the moment) the exponential map in $\overline{M}$. Show that $\tilde{M}_p$ is a smooth submanifold in some neighborhood of p, and has vanishing second fundamental form at p. (One might refer to $\tilde{M}_p$ as *totally geodesic at* p.) Show that if the second fundamental form of M, with respect to ξ, is positive definite then p has a neighborhood U in $\overline{M}$ in which

$$M \cap \tilde{M}_p \cap U = \{p\}.$$

Thus when the second fundamental form is definite, one might say that "M lies, locally, on one side of M_p" — a sort of local convexity. (NOTE: Understanding this exercise is the key to Exercise 4 in the first problem set.)

Definition. Let $M^{m-1} \subset \overline{M}^m$ be an immersed submanifold. We say a point $p \in M$ is *umbilic* if the second fundamental form of M at p is a scalar multiple of the first fundamental form.

Exercise 18. Let $\overline{M} = \mathbb{R}^m$. Show that if every point of M is umbilic, then M is a piece of a sphere in $\mathbb{R}^m$. In particular, if M is compact and everywhere umbilic in $\mathbb{R}^m$ then M is a sphere in $\mathbb{R}^m$.

Gradients and Hessians

Exercise 19. Let $f : M \to \mathbb{R}$ be a differentiable function on the Riemannian manifold M. The *gradient vector field of f on M*, $\operatorname{grad} f$, is defined by

$$\operatorname{grad} f = \theta^{-1}(df),$$

where df denotes the differential of f, and $\theta : TM \to TM^*$ denotes the natural bundle isomorphism given by

$$\theta(\xi)(\eta) = \langle \xi, \eta \rangle,$$

for all $p \in M$ and ξ, $\eta \in M_p$.

Assume that

$$|\operatorname{grad} f| = 1$$

on all of M. Show that the integral curves of $\operatorname{grad} f$ are geodesics.

Exercise 20. Show that for any Riemannian manifold M the distance function may be given analytically by

$$d(x, y) = \sup \{ |\psi(x) - \psi(y)| : \psi \in C^\infty, |\operatorname{grad} \psi| \le 1 \},$$

that is, where ψ varies over smooth functions for which $|\operatorname{grad} \psi| \le 1$ on all of M.

HINT: One checks that for such ψ one has

$$|\psi(x) - \psi(y)| \le d(x, y).$$

So the issue is to show that among these functions we may choose ψ so that $|\psi(x) - \psi(y)|$ is arbitrarily close to $d(x, y)$. To this end, given x and y, consider the function

$$\psi(z) = \min \{ d(z, x), d(y, x) \}.$$

$$|\psi(x) - \psi(y)| = d(x, y).$$

Check that ψ is uniformly Lipschitz, that

$$|\psi(z) - \psi(w)| \le d(z, w)$$

for *all* $z, w \in M$. Now one requires an argument that ψ may be approximated by C^∞ functions ψ_n for which $|\operatorname{grad} \psi_n| \le 1$.

Definition. Given a function f on a Riemannian manifold M, its first covariant derivative is, simply, its differential df.

The *Hessian of f*, $\operatorname{Hess} f$, is defined to be the second covariant derivative of f, that is, ∇df. So

$$(\operatorname{Hess} f)(\xi, \eta) = \xi(df(Y)) - (df)(\nabla_\xi Y),$$

where Y is any extension of η.

Exercise 21. Prove

(a) $\operatorname{Hess} f$ is symmetric in ξ, η;

(b) the $(1,1)$–tensor field associated with the $(0,2)$–field $\operatorname{Hess} f$ is given by

$$\xi \mapsto \nabla_\xi \operatorname{grad} f;$$

(c) a function with positive definite Hessian has no local maxima.

Exercise 22. Let $p \in M$ such that $f(p) = \alpha$, and $\operatorname{grad} f$ does not vanish at p. Then the level surface of f through p, $f^{-1}[\alpha]$, restricted to a sufficiently small neighborhood of p, is an embedded $(n-1)$–manifold. Show that the Hessian of f at p is given by

$$(\operatorname{Hess} f)_{|(f^{-1}[\alpha])_p} = \mathfrak{B}_{-\operatorname{grad} f_{|p}},$$

the second fundamental form of $f^{-1}[\alpha]$ associated to the normal vector $-\operatorname{grad} f$ at p.

Gauss–Bonnet and Poincaré–Hopf

Exercise 23. Consider an oriented Riemannian 2–manifold M with a local positively oriented frame field $\{e_1, e_2\}$ on a neighborhood U diffeomorphic to a subset of $\mathbb{R}^2$. Let X denote a vector field on M with isolated zero at $p \in U$, and let $\mathfrak{x}_1 = X/|X|$ be the associated unit vector field on $U \setminus \{p\}$. Let $\mathfrak{x}_2$ denote the vector field on $U \setminus \{p\}$ obtained by rotating $\mathfrak{x}_1$ by $\pi/2$ radians. Along any circle C (in local or polar coordinates) about p, one can write

$$\mathfrak{x}_1 = (\cos\theta)e_1 + (\sin\theta)e_2, \qquad \mathfrak{x}_2 = -(\sin\theta)e_1 + (\cos\theta)e_2.$$

Show that if ${\omega_j}^k$ denotes the connection 1–form of the frame field $\{e_1, e_2\}$ on U, and ${\tau_j}^k$ denotes the connection 1–form of the frame field $\{\mathfrak{x}_1, \mathfrak{x}_2\}$ on $U \setminus \{p\}$, then

$${\tau_1}^2 = d\theta + {\omega_1}^2.$$

Define

$$\text{index } X \text{ at } p := \frac{1}{2\pi}\int_C d\theta.$$

Exercise 24.

(a) Given an orientable compact Riemannian 2–manifold M with smooth vector field X whose set of zeros is the subset $\{p_1, \ldots, p_\ell\}$ of M. Show that

$$\int_M \mathcal{K}\, dA = 2\pi \sum_{j=1}^{\ell} (\text{index } X \text{ at } p_j).$$

This recaptures the Poincaré–Hopf theorem that the sum of the indices of the singularities of a vector field on a compact differentiable manifold is equal to its Euler characteristic.

Geodesics

Exercise 25. Let $\gamma : [0,1] \to M$ be a piecewise smooth path. Show that given $\epsilon > 0$, there exists $\delta > 0$ such that for any $t \in [0,1]$, any two points in the disk $B(\gamma(t) : \delta)$ are connected by a unique minimizing geodesic of length $< \epsilon$.

Exercise 26. Assume that M is Riemannian complete and noncompact. Then for every $p \in M$ there exists a *ray* emanating from p, that is, there exists a geodesic $\gamma : [0, +\infty) \to M$, $|\gamma'| = 1$, such that

$$d(\gamma(t), \gamma(s)) = |t - s| \qquad \text{for all} \;\; t, s \geq 0.$$

Exercise 27. Show that if M is a compact Riemannian manifold, then every nontrivial free homotopy class has a path of (positive) shortest length in the class, and this path is a closed geodesic.

Exercise 28. Assume $\gamma : [0, \beta] \to M$, $p = \gamma(0)$, $\xi = \gamma'(0)$, is a unit speed geodesic. Show that points along γ, conjugate to p along γ, are isolated.

Exercise 29. Let $p = \gamma(0)$, $\xi = \gamma'(0) \in \mathsf{S}_p$. For ϵ in $(0, 1)$, let $\mathcal{C}_\epsilon(\xi)$ denote the neighborhood of $\xi \in \mathsf{S}_p$ given by

$$\mathcal{C}_\epsilon(\xi) = \{\eta \in \mathsf{S}_p : \langle \xi, \eta \rangle > 1 - \epsilon\};$$

and for any ϵ in $(0, 1)$ and $r > 0$ let

$$\mathcal{C}_{\epsilon,r}(\xi) = \{t\eta \in M_p : t \in [0, r), \eta \in \mathcal{C}_\epsilon(\xi)\},$$

and

$$\mathfrak{C}_{\epsilon,r}(\xi) = \exp \mathcal{C}_{\epsilon,r}(\xi).$$

Prove the following

Theorem. *Assume $\gamma : [0, \beta] \to M$, $p = \gamma(0)$, $\xi = \gamma'(0)$, is a unit speed geodesic such that $\gamma|(0, \beta]$ is one-to-one with no points conjugate to p along γ. Then there exist ϵ in $(0, 1)$, $r > \beta$, such that $\mathcal{C}_{\epsilon,r}(\xi) \subseteq \mathcal{T}M$, the domain of the exponential map. Furthermore, there exists sufficiently small $\epsilon > 0$ such that, if ω is a path from p to $\gamma(\beta)$ with image completely contained in $\mathfrak{C}_{\epsilon,r}(\xi)$ then*

$$\ell(\omega) \geq \beta,$$

with equality only if the image of ω is the same as that of γ.

Exercise 30. Let M be a Riemannian manifold, p a point in M, and r *the distance function on M based at p*, that is, r is given by

$$r(x) = d(p, x).$$

(a) Show, for $r > 0$ sufficiently small, that $r \in C^\infty$ and $|\operatorname{grad} r| = 1$.

(b) Show, with $\beta > 0$ sufficiently small as in (a), $\gamma : [0, \beta] \to M$ a unit speed geodesic emanating from p, that for any Jacobi field Y along γ, vanishing at p and orthogonal to γ along γ, its index form I is given by

$$I(Y, Y) = \langle \nabla_t Y, Y \rangle(\beta) = \mathfrak{B}_{-\operatorname{grad} r_{|\gamma(\beta)}}(Y(\beta), Y(\beta)) = \operatorname{Hess} r(Y(\beta), Y(\beta))$$

where $\mathfrak{B}$ denotes the second fundamental form of the level surface $r^{-1}[\beta]$.

Riemann normal coordinates

Exercise 31. Given $p \in M$, $\xi, \eta, \zeta \in M_p$, $|\xi| = 1$, $\gamma(t) = \exp t\xi$, and Y, Z Jacobi fields along γ determined by

$$\begin{aligned} Y(0) &= 0, \qquad (\nabla_t Y)(0) = \eta, \\ Z(0) &= 0, \qquad (\nabla_t Z)(0) = \zeta; \end{aligned}$$

then the Taylor expansion of $\langle Y, Z\rangle(t)$ about $t = 0$ is given by

$$\langle Y, Z\rangle(t) = t^2\langle \eta, \zeta\rangle - (t^4/3)\langle R(\xi, \eta)\xi, \zeta\rangle + 0(t^5).$$

HINT: Direct calculation. The idea is that Taylor's expansion, in a neighborhood of $t = 0$, for a vector field $Y(t)$ along the geodesic $\gamma_\xi(t)$ is given by

$$Y(t) = \tau_t \left\{Y(0) + t\nabla_t Y(0) + (t^2/2){\nabla_t}^2 Y(0) + (t^3/6){\nabla_t}^3 Y(0)\right\} + O(t^4),$$

where τ_t denotes parallel translation along γ_ξ from p to $\gamma_t(\xi)$. Now one uses the hypotheses of the theorem to calculate the derivatives of $Y(t)$ at $t = 0$. In the inner product, one uses the fact that the parallel translation is an isometry.

Exercise 32. Assume $\dim M = 2$, $p \in M$, $L(r)$ the length of $S(p; r)$, and $A(r)$ the area of $B(p; r)$, $K(p)$ the Gauss curvature of M at p. Show that

$$K(p) = \lim_{r\downarrow 0} \frac{2\pi r - L(r)}{\pi r^3/3} = \lim_{r\downarrow 0} \frac{\pi r^2 - A(r)}{\pi r^4/12}.$$

HINT: Apply previous exercise to geodesic polar coordinates about p.

Exercise 33. Fix $p \in M$ and U an open set about, and starlike with respect to, $0 \in M_p$ for which $\exp|\mathsf{U}$ is a diffeomorphism of U onto its image $U := \exp \mathsf{U}$, an open set in M about p.

Then every choice of orthonormal basis $\{e_1, \ldots, e_n\}$ of M_p determines a chart $\mathbf{n} : U \to \mathbb{R}^n$, referred to as *Riemann normal coordinates,* given by

$$\mathbf{n}^j(q) = \langle(\exp|\mathsf{U})^{-1}(q), e_j\rangle$$

for $q \in U$, that is, for $v = \sum_j v^j e_j \in \mathsf{U}$ we have

$$\mathbf{n}^j(\exp v) = v^j.$$

In this chart we have for $\gamma(t) = \exp tv$,

$$\gamma^j(t) := (\mathbf{n}^j \circ \gamma)(t) = tv^j, \qquad \gamma'(t) = \sum_j v^j \partial_{j|\gamma(t)}.$$

(a) Let Y_j be the Jacobi field along γ determined by the initial conditions

$$Y_j(0) = 0, \quad (\nabla_t Y_j)(0) = e_j.$$

prove that

$$\partial_{j|\exp tv} = (\exp_p)_{*|tv} \Im_{tv} e_j = t^{-1} Y_j(t)$$

for $tv \in \mathsf{U}$.

(b) Prove that, for $v \in \mathsf{U}$,

$$g_{jk}(\exp v) = \delta_{jk} - (1/3)\langle R(v, e_j)v, e_k\rangle + O(|v|^3),$$

and

$$\det(g_{jk}(\exp v)) = 1 - (1/3)\mathrm{Ric}\,(v, v) + O(|v|^3),$$

as $v \to 0$.

Exercise 34. Given a Riemannian manifold M, $p \in M$. Show that given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{d(\exp \xi, \exp \eta)}{|\xi - \eta|} = 1 \pm O(\epsilon^2)$$

for all $\xi, \eta \in \mathsf{B}(p; \delta)$.

Hyperbolic space

Exercise 35. Given the unit n–disk $\mathbb{B}^n$ with the Riemannian metric

$$ds^2 = \frac{4|dx|^2}{\{1 - |x|^2\}^2}.$$

Use orthonormal moving frames to show that the sectional curvature is identically equal to -1. (HINT: Use the moving frame defined by

$$E_A = \frac{1 - |x|^2}{2} e_A, \qquad \omega^A = \frac{2}{1 - |x|^2}\, dx^A,$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.)

The Myers–Steenrod Theorem

Exercise 36. Let M be a Riemannian manifold, and $\varphi : M \to M$ an onto map (not assumed to be continuous) such that $d(\varphi(p), \varphi(q)) = d(p, q)$ for all $p, q \in M$. Then φ is an isometry, that is, φ is a diffeomorphism preserving the Riemannian metric.

SKETCH:

(a) Show that φ is a homeomorphism.

(b) Fix p and $\varphi(p)$ in M. Let δ_1 denote the injectivity radius of M at p. Show that we have a well defined map $F : \mathsf{B}(p; \delta_1) \to \mathsf{B}(\varphi(p); \delta_1)$ defined by

$$F(\xi) = (\exp|\mathsf{B}(\varphi(p); \delta_1))^{-1} \circ \varphi \circ \exp \xi.$$

(c) Show that for $\xi \in \mathsf{B}(p; \delta_1)$, $s \in [0, 1]$ one has

$$F(s\xi) = sF(\xi). \tag{1}$$

Then show that F may be extended to all of M_p so that it satisfies (1) and

$$|F(\xi)| = |\xi| \tag{2}$$

for all $\xi \in M_p$, $s \geq 0$.

(d) Next, use Exercise 13 to show that, given any $\epsilon > 0$, there exists sufficiently small $\delta > 0$ so that

$$|F(\xi) - F(\eta)| = |\xi - \eta|\{1 \pm O(\epsilon^2)\}$$

for all $\xi, \eta \in \mathsf{B}(p;\delta)$.

(e) Next, show $|F(\xi) - F(\eta)| = |\xi - \eta|$ for all $\xi, \eta \in M_p$.

(f) Let $|\xi| = |\eta| = 1$. Use the formula

$$|\xi - \eta| = 2 \sin \frac{1}{2}\measuredangle(\xi, \eta)$$

and (2) to show that

$$\sin \frac{1}{2}\measuredangle(F(\xi), F(\eta)) = \sin \frac{1}{2}\measuredangle(\xi, \eta),$$

which therefore implies

$$\cos \measuredangle(F(\xi), F(\eta)) = \cos \measuredangle(\xi, \eta),$$

which implies F preserves the inner product.

(g) Use the expansion of vectors with respect to an orthonormal basis of an inner product space to show that F is additive, and therefore, linear. Then show that $F = \varphi_{*|p}$, which implies the lemma.

Hopf fibration of $\mathbb{S}^3$

Exercise 37. We let $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the standard basis of $\mathbb{R}^4$. Beyond the vector space structure of $\mathbb{R}^4$ we define a multiplication of elements of $\mathbb{R}^4$, where the multiplication of the natural basis is given by

$$\mathbf{1i} = \mathbf{i} = \mathbf{i1}, \quad \mathbf{1j} = \mathbf{j} = \mathbf{j1}, \quad \mathbf{1k} = \mathbf{k} = \mathbf{k1}, \qquad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1},$$

and

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

With this bilinear multiplication, $\mathbb{R}^4$ becomes an algebra, the *quaternions.*

With each element

$$\mathbf{x} = \alpha\mathbf{1} + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$$

we associate its *conjugate*

$$\overline{\mathbf{x}} = \alpha\mathbf{1} - \beta\mathbf{i} - \gamma\mathbf{j} - \delta\mathbf{k},$$

and its *norm* $|\mathbf{x}|$ defined by

$$|\mathbf{x}|^2 := \mathbf{x}\overline{\mathbf{x}} = \overline{\mathbf{x}}\mathbf{x}$$

with associated bilinear form

$$\langle \mathbf{x}|\mathbf{y}\rangle = \frac{1}{2}(\overline{\mathbf{x}}\mathbf{y} + \overline{\mathbf{y}}\mathbf{x}).$$

Note that $\overline{\mathbf{xy}} = \overline{\mathbf{y}}\overline{\mathbf{x}}$, which implies

$$|\overline{\mathbf{x}}|^2 = |\mathbf{x}|^2, \qquad |\mathbf{xy}| = |\mathbf{x}||\mathbf{y}|.$$

So

$$|\mathbf{x}| = 1 \qquad \Rightarrow \qquad |\mathbf{xy}| = |\mathbf{yx}| = |\mathbf{y}|.$$

We conclude that the unit quaternions, $\mathbb{S}^3$ is a compact Lie group under the quaternionic multiplication.

Also, for any $\mathbf{x} \neq 0$ we have

$$\mathbf{x}^{-1} = \frac{\overline{\mathbf{x}}}{|\mathbf{x}|^2}.$$

Since $\mathbf{1}$ is the identity element of the unit quaternions, the basis of the tangent space to $\mathbb{S}^3$ at $\mathbf{1}$ can be thought of as given by

$$\mathbf{i}, \quad \mathbf{j}, \quad \mathbf{k}.$$

More precisely, it is given by

$$\Im_{\mathbf{1}}\mathbf{i}, \quad \Im_{\mathbf{1}}\mathbf{j}, \quad \Im_{\mathbf{1}}\mathbf{k}.$$

(a) Show that if $\boldsymbol{\xi}$ is a linear combination of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ then (i) $\boldsymbol{\xi}^2 = -|\boldsymbol{\xi}|^2$, and (ii) the 1–parameter subgroup in $\mathbb{S}^3$, $\gamma(t) = \exp t\boldsymbol{\xi}$, is given by

$$\exp t\boldsymbol{\xi} = (\cos |\boldsymbol{\xi}|t)\mathbf{1} + (\sin |\boldsymbol{\xi}|t)\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}.$$

(b) Show that the Lie algebra is given by

$$[\mathbf{i}, \mathbf{j}] = 2\mathbf{k}, \quad [\mathbf{j}, \mathbf{k}] = 2\mathbf{i}, \quad [\mathbf{k}, \mathbf{i}] = 2\mathbf{j}.$$

(c) Declare the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of $\mathbb{S}^3$ to be orthonormal in the tangent space to $\mathbb{S}^3$ at $\mathbf{1}$ ($\mathbb{S}^3$ does not have a Riemannian metric, yet), and use left-invariance to define a Riemannian metric on $\mathbb{S}^3$. Show that the Riemannian metric is bi-invariant, and has sectional curvature identically equal to 1.

(d) Let H denote the Lie subgroup

$$H = \mathbb{S}^1 = \{\cos\theta\mathbf{1} + \sin\theta\mathbf{i} : \theta \in \mathbb{R}\}.$$

Show that G/H is the 2–sphere in $\mathbb{R}^3$ with constant sectional curvature equal to 4.

The orthogonal group

Exercise 38. Given the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

determine the 1–parameter subgroup of $O(2)$ with A its initial velocity vector at the identity.

As an exercise, do it by literally calculating e^{tA}.

Exercise 39. Recall that $\mathfrak{M}^n$ denotes the space of $(n \times n)$ real matrices, and $GL(n)$ the nonsingular elements of $\mathfrak{M}^n$. Let $\mathrm{Sym}\,(n)$ denote the symmetric $(n \times n)$ matrices.

(a) Prove that the inclusion map into $\mathrm{Sym}\,(n) \to \mathfrak{M}^n$ realizes $\mathrm{Sym}\,(n)$ as an $n(n+1)/2$–dimensional submanifold of $\mathfrak{M}^n$.

(b) Show that the mapping

$$\psi : GL(n) \to \mathrm{Sym}\,(n), \qquad \psi(A) = A \cdot A^T,$$

where A^T denotes the transpose of A, has the unit matrix I for a regular value.

(c) Show that the orthogonal group $O(n) = \psi^{-1}[I]$ is an $n(n-1)/2$–dimensional compact submanifold of $GL(n)$.

(d) Show that the tangent space to $O(n)$ at the identity may be represented by the skew-symmetric $(n \times n)$ matrices.

(e) Prove that $O(n)$ has precisely two components.

Exercise 40.

(a) Show that $SO(3)$ is diffeomorphic to $\mathbb{RP}^3$ (3–dimensional real projective space).

(b) What about the unit tangent bundle of $\mathbb{S}^2$?

Coverings and bundles

Exercise 41. Show that if $\pi : M \to M_o$ is a differentiable covering, M orientable, and the deck tranformation group of the covering has an orientation reversing diffeomorphism, then M_o is nonorientable.

Exercise 42. Let $\pi : M \to M_o$ be a Riemannian covering, M compact. Characterize $V(M)/V(M_o)$.

Definition. Recall that an *n–dimensional vector bundle* consists of a projection

$$\pi : E \to B$$

of E *onto* B, where E and B are C^∞ manifolds, $\pi \in C^\infty$, such that:

(i) for each $q \in B$, the fiber $\pi^{-1}[q]$ has the structure of n–dimensional vector space;

(ii) *local triviality*: for each $p \in B$ there exists a neighborhood $U = U(p)$ and a homeomorphism $\phi : \pi^{-1}[U] \to U \times \mathbb{R}^n$ such that $\phi|\pi^{-1}[q]$ is an isomorphism of $\pi^{-1}[q]$ to $\mathbb{R}^n$.

We say that two vector bundles $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ are *equivalent* if there exist C^∞ diffeomorphisms $\Phi : E_1 \to E_2$ and $\phi : B_1 \to B_2$ such that for every $p \in B_1$, $\Phi|\pi_1^{-1}[p]$ is a vector space isomorphism of ${\pi_1}^{-1}[p]$ to ${\pi_2}^{-1}[\phi(p)]$.

We say that the projection $\pi : M \times \mathbb{R}^n \to M$ onto the first factor is *the standard trivial* bundle. We refer to a vector bundle as *trivial* if it is equivalent to the standard trivial bundle.

Exercise 43. Consider the subgroup $\mathcal{T} = \{T^n : \ n \in \mathbb{Z}\}$ of diffeomorphisms of $\mathbb{R}^2$ generated by the mapping

$$T : (x, y) \mapsto (x + 1, -y).$$

Show that the quotient space $E = \mathbb{R}^n/\mathcal{T}$ is a 1–dimensional vector bundle over $\mathbb{S}^1$, but is not trivial.

Degree theory

Exercise 44. Show that the degree of composition $g \circ f$ is equal to the product $(\deg g)(\deg f)$.

Exercise 45. Show that every complex polynomial of degree n gives rise to a map of the sphere $\mathbb{S}^2$ to itself of degree n.

Exercise 46. Show that if two maps f and g from the manifold X to $\mathbb{S}^n$ satisfy $|f(x)-g(x)| < 2$ for all $x \in X$, then f is homotopic to g, the homotopy being smooth if f and g are smooth.

Exercise 47. Show that any map $\mathbb{S}^n$ to $\mathbb{S}^n$ of odd degree must carry some pair of antipodal points into a pair of antipodal points.

Exercise 48. Let ω be a 1–form on $\mathbb{S}^2$ invariant under all orthogonal transformations of $\mathbb{R}^3$. Show that ω must vanish identically.