Differential Geometry Qualifying Exam
Fall 2017

Do any **6 problems**.

Note: Throughout this exam, all manifolds are C^∞ and connected, and all maps are C^∞ unless it is specifically stated otherwise.

1. Consider $S^n \subset \mathbb{R}^{n+1}$ with the induced metric and let $p = (1, 0, \ldots, 0, 0) \in S^n$. Find an explicit expression of the exponential map $\exp_p : T_pS^n \to S^n$. Prove that it has maximal rank except at the points $\vec{v} \in T_pS^n$ with $|\vec{v}| = \pm k\pi$, $k \in \mathbb{N}$, where it has rank 1.

2. Consider a mapping ϕ from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by $\phi(u, v) = (u+v, uv, 2)$. Compute $\phi^*(dx)$, $\phi^*(dy)$, $\phi^*(dz)$, $\phi^*(ydx)$, and $\phi^*(zdy)$, where (x, y, z) are the coordinates in $\mathbb{R}^3$.

3. Let G be a Lie group endowed with a bi-invariant metric. Prove that the map $\psi : G \to G$ given by $\psi(g) = g^{-1}$ is an isometry.

4. Let M be a Riemannian manifold and $p \in M$. For some normal ball $B_r(p) \subset M$, let $s_p : B_r(p) \to B_r(p)$ be the local diffeomorphism defined by
$$s_p(\exp_p(\vec{v})) = \exp_p(-\vec{v}).$$
Suppose that s_p is an isometry in some geodesic ball $B_r(p)$. Prove that $(\nabla R)_p = 0$, where R is the Riemann curvature tensor.

5. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere centered at the origin. Show that if two maps f and g from a manifold M to S^n satisfy
$$|f(x) - g(x)| < 2$$
for all $x \in M$, then f is homotopic to g, the homotopy being smooth if f and g are smooth.

6. Let M be a smooth manifold of dimension n and let $\omega, \eta \in \Omega^n(M)$ be two differential forms of top degree that do not vanish at any point of M. Prove that there exists a C^∞ function f on M such that $\omega = f\eta$.

7. Let X, Y be smooth vector fields on a manifold M and let α be a curve with $\alpha(0) = p \in M$ and $\alpha'(0) = X_p$. Let ∇ denote the Levi-Civita connection on M and let $P_{0,t}$ denote parallel translation along α from $\alpha(0)$ to $\alpha(t)$. Prove that
$$(\nabla_X Y)_p = -\lim_{t \to 0} \frac{P_{0,t}(Y_p) - Y_{\alpha(t)}}{t}.$$

8. Consider the action of the group $SL(2,\mathbb{R})$ on the upper half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ given by

$$gz = \frac{az+b}{cz+d},$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Compute the isotropy subgroup of i and use this to identify the upper half plane with a quotient of $SL(2,\mathbb{R})$.

9. Let M be a complete Riemannian manifold, $p \in M$, and let $\vec{v}, \vec{w} \in T_pM$. Let $\gamma(t) = \exp_p(\vec{v}t)$ be a geodesic. Prove that

$$J(t) = (d\exp_p)_{t\vec{v}}(t\vec{w})$$

is a Jacobi field along γ with $J(0) = 0$.

10. Let ω be a 1–form on S^2 invariant under all orthogonal transformations of $\mathbb{R}^3$. show that ω must vanish identically.

11. Let M be a 2-dimensional manifold with metric given locally, in a coordinate patch, by $g = e^{2f}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$, where $f : M \to \mathbb{R}$ is a C^∞ function. Show that the sectional curvature K is given by

$$K = -e^{-2f}\Delta f,$$

where Δ is the usual Laplacian given by $\Delta f = \dfrac{\partial^2 f}{\partial x_1^2} + \dfrac{\partial^2 f}{\partial x_2^2}$.