DIFFERENTIAL GEOMETRY QUALIFYING EXAM
FALL 2021

Instructions: No more than 6 problems will be graded—specify which ones you want graded.
Note: Throughout this exam, all manifolds are $\mathcal{C}^\infty$ and connected, and all maps are $\mathcal{C}^\infty$ unless it is specified otherwise.

(1) Let $p : \mathbb{R}^{n+k} \to \mathbb{R}^n, (x_1, \ldots, x_{n+k}) \mapsto (x_1, \ldots, x_n)$ be the projection map. Prove that the forms on $\mathbb{R}^{n+k}$ that are pullbacks of forms on $\mathbb{R}^n$ are exactly those that are in the kernel of the interior multiplications by $\frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_{n+k}}$ and whose d is also in the kernel of the interior multiplications by $\frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_{n+k}}$.

(2) Let α be a 1-form in $\mathbb{R}^3$. Show that if α is invariant under all isometries of $\mathbb{R}^3$, then α must be zero.

(3) For $n \geq 1$ denote by $S^n \subset \mathbb{R}^{n+1}$ the n-sphere, and define $f : S^n \to S^n, f(x) := -x$ to be the antipodal map. Show that f is orientation-preserving iff n is odd.

(4) Let $(M, g = \langle , \rangle)$ be a Riemannian manifold with Levi-Civita connection ∇. For a function $f : M \to \mathbb{R}$, the gradient of f is the vector field $\mathrm{grad}(f) \in \mathfrak{X}(M)$ defined by

$$\langle \mathrm{grad}(f), X \rangle = df(X) = X(f) \qquad \forall X \in \mathfrak{X}(M)$$

For a vector field $V \in \mathfrak{X}(M)$, the $\mathrm{curl}(V)$ is the $(0,2)$-tensor field defined by

$$\mathrm{curl}(V)(X, Y) = \langle \nabla_X V, Y \rangle - \langle \nabla_Y V, X \rangle \qquad \forall X, Y \in \mathfrak{X}(M)$$

Show that

$$\mathrm{curl}(\mathrm{grad}(f)) = 0$$

(5) Let M be the surface $M := \{(s, t, s \cdot t) \mid s, t \in \mathbb{R}\}$ in $\mathbb{R}^3$.

(a) Show that at $p = (s, t, s \cdot t) \in M$, the vectors $(1, 0, t) \in T_p\mathbb{R}^3$ and $(0, 1, s) \in T_p\mathbb{R}^3$ are tangent vectors of M, and, furthermore, that $N = \frac{(t,s,-1)}{\sqrt{1+s^2+t^2}} \in T_p\mathbb{R}^3$ is a unit normal vector to M.

(b) Show that the Gaussian curvature of M is

$$K = \frac{-1}{(1 + s^2 + t^2)^2}$$

(6) Consider the metric on $\mathbb{R}^2$ given by

$$dx \otimes dx + (1 + x^2)^2 \cdot dy \otimes dy$$

Find the Gaussian curvature of this metric.

(7) Denote by $G := \{A \in GL(n,\mathbb{R}) \mid \exists c > 0 : A^{-1} = c \cdot A^t\}$.
   (a) Show that G with the matrix multiplication is a Lie group by showing that it is isomorphic as a Lie group to $(O(n), \cdot) \times (\mathbb{R}, +)$.
   (b) Identify the Lie algebra $\mathfrak{g}$ of G as a sub-Lie algebra of $\mathfrak{gl}(n,\mathbb{R})$.

(8) Let $\{U_i\}_{i\in I}$ be an open cover of a manifold M, let V be a vector space, and let $\{g_{ij} : U_i \cap U_j \to Aut(V)\}_{ij}$ be the transition functions for a vector bundle E with fiber V. Furthermore, let $\{A_i \in \Omega^1(U_i, End(V))\}_i$ be the 1-forms of a connection on E, i.e., the A_i satisfy

$$A_i = g_{ji}^{-1} \cdot A_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} \qquad \text{on } U_i \cap U_j.$$

If $\{f_i : U_i \to Aut(V)\}_i$ are such that for all i, j, the identity $g_{ji} \cdot f_i = f_j \cdot g_{ji}$ holds on $U_i \cap U_j$, then show that

$$B_i := f_i^{-1} \cdot A_i \cdot f_i + f_i^{-1} \cdot df_i$$

are also the 1-forms of a connection on E, i.e., they satisfy

$$B_i = g_{ji}^{-1} \cdot B_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} \qquad \text{on } U_i \cap U_j.$$

(9) Let $E \to M$ be a smooth vector bundle with affine connection ∇. Denote by Ω its curvature 2-form. Show that the trace of the curvature, $tr(\Omega)$, is a closed, globally defined 2-form.

Hint 1: You may use without proof that d commutes with the trace.

Hint 2: You may use without proof that for a matrix of p-forms A and a matrix of q-forms B, we have $tr(A \wedge B) = (-1)^{p\cdot q} \cdot tr(B \wedge A)$.

(10) Let (M, g) be a Riemannian manifold. For a given chart of M denote by g_{ij} the components of the metric tensor g, and denote by Γ^k_{ij} the Christoffel symbols of the Levi-Civita connection. Prove the following identities:
   (a) $\Gamma^k_{ij} = \Gamma^k_{ji}$
   (b) $\frac{\partial g_{ij}}{\partial x^k} = g_{\ell j}\Gamma^\ell_{ki} + g_{i\ell}\Gamma^\ell_{kj}$
   (c) Use (a) and (b) to show: $\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \cdot \left(\frac{\partial g_{i\ell}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right)$

(11) Let T be a regular tetrahedron.
   (a) Calculate the Euler characteristic of T.
   (b) Confirm the combinatorial Gauss-Bonnet theorem for T.