Differential Geometry Qualifying Exam
Spring 2011

Instructions: No more than five problems will be graded—specify which ones you want graded.

Problem 1. Let $(M, g)$ and $(M', g')$ be $n$-dimensional Riemannian manifolds.

(a) Define what it means for $(M, g)$ and $(M', g')$ to be isometric.

(b) Define what it means for $(M, g)$ and $(M', g')$ to be conformally equivalent.

(c) Let

$$M = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1, z > 0\}$$

and

$$M' = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = z^2, z > 0\},$$

each with metric induced by the standard Euclidean metric in $\mathbb{R}^3$. Prove or disprove: $M$ and $M'$ are isometric.

Problem 2. Give a precise definition of a smooth vector bundle.

(a) Define what it means for a smooth vector bundle to be trivial.

(b) Give an example of a smooth vector bundle that is not trivial and prove that it is not trivial.

Problem 3. Prove that the set $\{(x, y) \in \mathbb{R}^2 | x^2 = y^3\}$ is not a smooth submanifold of $\mathbb{R}^2$.

Problem 4. Let $M$ be a smooth hypersurface in $\mathbb{R}^n$. Prove that $M$ is orientable if and only if there exists a transverse smooth vector field along $M$. (For this problem, you may use any intrinsic definition of orientability.)

Problem 5. Consider vector fields on $\{(x, y, z) \in \mathbb{R}^3 : x, y, z > 0\}$ given by

$$X = xy \frac{\partial}{\partial x} - yz \frac{\partial}{\partial z} \quad \text{and} \quad Y = x \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial y}.$$ 

(a) Let $\phi_t$ and $\psi_s$ be the flows generated by $X$ and $Y$ respectively and compute $\phi_t(x_0, y_0, z_0)$ and $\psi_s(x_0, y_0, z_0)$.

(b) Do $X$ and $Y$ span an involutive distribution? Explain.

Problem 6. Show that the smooth manifold $S^1 \times S^2$ does not admit a metric whose sectional curvatures are all positive.
Problem 7. Let $\Sigma$ be a smooth surface of genus equal to 2.

(a) Draw the image of an embedding $\phi: \Sigma \to \mathbb{R}^3$. Label at least one point of positive and negative curvature, respectively.

(b) Prove that for any embedding $\phi$ there is an open subset of $\phi(\Sigma)$ where curvature is negative.

Problem 8. Consider the surface

$$M = \{(x, y, z) \in \mathbb{R}^3 | x = y^2 + z^2 \leq 4\}$$

oriented in such a way that the basis $\left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right\}$ of $T_{(0,0,0)} M$ is positively oriented. Compute:

(a) $\int_M xy^2 \, dx \wedge dz + \sin(yz^2) \, dy \wedge dz$

(b) $\int_{\partial M} xz \, dy + y^2 \, dz$.

Problem 9. Let $G$ be a connected Lie group with bi-invariant metric $g$. Let $\nabla$ denote the Levi-Civita connection. Then for left invariant vector fields $X, Y$ and $Z$,

$$g ([X, Y], Z) = g (X, [Y, Z]) \quad \text{and} \quad \nabla_X Y = \frac{1}{2} [X, Y].$$

(a) Show that the curvature tensor of $\nabla$ satisfies

$$Rm(X, Y, Z, W) = -\frac{1}{4} g ([X, Y], [Z, W])$$

whenever $X, Y, Z$ and $W$ are left invariant vector fields.

(b) Show that the sectional curvature of $(G, g)$ is non-negative.

(c) Prove that the Lie algebra $\mathfrak{g}$ of $G$ has zero bracket if and only if the sectional curvature of $(G, g)$ is zero.