Differential Geometry Qualifying Exam
Spring 2012

Instructions: No more than *six* problems will be graded—*specify which ones you want graded.*

Problem 1. Let $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be the trivial line bundle over $\mathbb{R}^2$. Note (by a slight abuse of notation) that a section s is a function $s : \mathbb{R}^2 \to \mathbb{R}$.

(a) Prove that for a real valued 1-form ω on $\mathbb{R}^2$, the formula

$$\nabla_X(s) = ds(X) + \omega(X)s \text{ for a vector field } X \text{ on } \mathbb{R}^2 \text{ and a section } s$$

determines a connection on this bundle.

(b) Consider the connection ∇ determined by the 1-form $\omega = \frac{1}{2}(xdy - ydx)$. Prove that parallel translation along a simple closed curve $\gamma : [0,1] \to \mathbb{R}^2$ acts on the fiber over $\gamma(0)$ by

$$v \mapsto \exp\left(\int_\gamma \omega\right) v.$$

(c) Show that the log of the parallel translation around a simple closed curve γ equals the area enclosed by γ.

Problem 2. Consider the unit sphere in $\mathbb{R}^3$ with standard coordinates (x, y, z), and let M be the part of the unit sphere with $x > 0$. Consider stereographic coordinates on M given by $\varphi : (0, \infty) \times \mathbb{R} \longrightarrow M$ where

$$\varphi(u, v) = \frac{(2u, 2v, u^2 + v^2 - 1)}{u^2 + v^2 + 1}.$$

Note that M can also be regarded as a graph over the (y, z) coordinates.

(a) Compute the transitions map between the (u, v) coordinates for M and the (y, z) coordinates for M.

(b) Rewrite $\dfrac{\partial}{\partial v}$ and du in (y, z) coordinates.

Problem 3. Let N be an embedded submanifold of a manifold M. Assume that N is closed in M. Prove that for every smooth form $\omega \in A^1(N)$ of degree one, there exists a smooth form $\tilde{\omega} \in A^1(M)$ such that $i^*\tilde{\omega} = \omega$ where $i : N \longrightarrow M$ is the inclusion map. Is the same true for forms of degree other than one? Is the same true if N is not closed in M?

Problem 4. Let $G = SO(n, \mathbb{R})$ be the group of orthogonal matrices A with $\det A = 1$.

(a) Prove that $SO(n, \mathbb{R})$ is connected.

(b) Compute the Lie algebra of G and the dimension of G.

Problem 5. Let M be a compact, smooth, oriented Riemannian manifold with boundary. Prove that there does not exist a smooth retraction of M onto its boundary.

Problem 6. If M is a complete Riemannian manifold and $N \subset M$ a closed, embedded submanifold with the induced Riemannian metric, show that N is complete. *Warning:* The distance function on N induced from the metric space structure of M is *not* in general equal to the Riemannian distance function of N.

Problem 7. Show that the smooth manifold $S^3 \times S^2$ does not admit a metric whose sectional curvatures are all negative.

Problem 8. Let (M, g) and (M', g') be n-dimensional Riemannian manifolds.

(a) Define a *conformal equivalence* between (M, g) and (M', g').

(b) Define an *isometry* between (M, g) and (M', g').

(c) Let

$$M = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1, z > 0\}$$

and

$$M' = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = z^2, z > 0\}$$

each with metric induced by the standard Euclidean metric in $\mathbb{R}^3$. Is $f : M \to M'$, given by $f(x, y, z) = (xz, yz, z)$ a conformal equivalence? Justify your answer.

Problem 9. Let Σ be a smooth closed surface of genus $g > 0$. Let $\phi : \Sigma \to \mathbb{R}^3$ be any smooth embedding.

(a) Define the *second fundamental form.*

(b) Show that the second fundamental form is not everywhere positive definite, nor it is it everywhere negative definite.

Problem 10. Consider $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ defined by

$$F(u, v) = (u^2 + v^2, u + v, 2uv, u - v).$$

(a) Prove that F is a smooth embedding.

(b) Let M be the image of F. Given a point $(x, y, z, w) \in M$, find a basis for $T_{(x,y,z,w)}M$ in (x, y, z, w) coordinates.

Problem 11. Let M be a surface in $\mathbb{R}^3$, and let $\mathbf{x}_0$ be a point in M that maximizes the value of $\|\mathbf{x}\|$ over M. Prove that M has nonnegative Gauss curvature at $\mathbf{x}_0$.

Problem 12. Let G be a connected Lie group with bi-invariant metric g and let X, Y, and Z be left invariant vector fields.

(a) Prove that $g([X, Y], Z) = g(X, [Y, Z])$

(b) Prove that the Levi-Civita connection satisfies $\nabla_X Y = \frac{1}{2}[X, Y]$.

(c) Show that the curvature tensor of this connection satisfies $R(X, Y, Z) = \frac{1}{4}[Z, [X, Y]]$.