DIFFERENTIAL GEOMETRY QUALIFYING EXAM
SPRING 2014

Instructions: No more than 6 problems will be graded—specify which ones you want graded.
Note: Throughout this exam, all manifolds are C^∞ and connected, and all maps are C^∞ unless it is specified otherwise.

(1) Let $F : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^3$ be defined by $F(u,v) = (v, u+v, \sqrt{u^2+v^2})$. Prove that the image of F is a smooth embedded submanifold of $\mathbb{R}^3$.

(2) Consider the 2-form α on $\mathbb{R}^3 - \{0\}$ given by
$$\alpha = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}.$$
  (a) Show that α is closed.
  (b) Show that α is not exact.
  (c) Show that α restricted to the half-space $\{(x,y,z) \in \mathbb{R}^3 | x > 0\}$ is exact.

(3) Let $V = x\frac{\partial}{\partial x}$ and $W = x\frac{\partial}{\partial y} + \frac{\partial}{\partial x}$ be two vector fields on $\mathbb{R}^2$. Verify the formula for the Lie derivative with the following steps.
  (a) Calculate the Lie bracket $[V, W]$.
  (b) Find the flow $\theta_t : \mathbb{R}^2 \to \mathbb{R}^2$ of V (*i.e.* find the integral curve $\gamma_{(x,y)}(t) = \theta_t(x,y)$ with $\gamma'_{(x,y)}(t) = V_{\gamma_{(x,y)}(t)}$ and $\gamma_{(x,y)}(0) = (x,y)$.)
  (c) Calculate $(d\theta_{-t})_{\theta_t(a)}(W_{\theta_t(a)})$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.
  (d) Calculate $\mathcal{L}_V W = \lim_{t\to 0} \frac{(d\theta_{-t})(W)-W}{t}$ and check that it coincides with $[V,W]$ as calculated in (a).

(4) Let M be a manifold of dimension d. Prove or give a counterexample for the following statements.
  (a) If $a \in M$ and $v_1, \ldots, v_k \in T_aM$, are k tangent vectors at a, then there exists a smooth map $f : U \to M$ with $0 \in U \subset \mathbb{R}^k$ open, such that $f(0) = a$ and $df_0(\frac{\partial}{\partial x^j}|_0) = v_j$ for all $j = 1, \ldots, k$.
  (b) If $V \subset M$ is an open subset and $X_1, \ldots X_k : V \to TM$ are k pointwise linearly independent smooth vector fields on V, then there exists an immersed submanifold $N \subset M$ such that $span(X_1|_a, \ldots, X_k|_a) = T_aN$ for all $a \in N$.

(5) Prove that for any smooth vector bundle E over a manifold M, there always exists a smooth global section of $E^* \otimes E^*$ that is a positive definite symmetric bilinear form on E_p at each point $p \in M$.

(6) For each question below, explain your reasoning thoroughly. In the last two parts, the term "parallel" means with respect to the Levi-Civita connection.
  Which compact surfaces admit a metric with the property that:
  (a) every point has a neighborhood with an orthonormal frame?
  (b) some point has a neighborhood with a parallel orthonormal frame?
  (c) every point has a neighborhood with a parallel orthonormal frame?

(7) Let $\pi : M \to N$ be a submersion. Let $\omega \in \Omega^k(M)$ be a differential form. Show that if $\omega(X_1, \ldots, X_k) = 0$ whenever $X_1, \ldots, X_k \in Ker(\pi_*)$, then $(d\omega)(Y_1, \ldots, Y_{k+1}) = 0$ whenever $Y_1, \ldots, Y_{k+1} \in Ker(\pi_*)$.

(8) Let G be a Lie group with bi-invariant metric $\langle , \rangle$, and Lie algebra $(\mathfrak{g}, [,])$. Then the induced Levi-Civita connection satisfies $\nabla_X Y = \frac{1}{2}[X, Y]$ for left invariant vector fields X and Y.
  (a) Define the exponential map from the Lie algebra $\mathfrak{g}$ to G.
  (b) Define the exponential map of a Riemanniann manifold (M, g).
  (c) Show that for the Riemannian manifold $(G, g = \langle , \rangle)$, these two exponential maps defined at the identity agree with each other.
  (d) Show every Lie group with bi-invariant metric is a complete Riemannian manifold.

(9) Let $\pi : E \to M$ be a vector bundle with connection ∇. Suppose that for some $p \in M$, the holonomy satisfies $hol(\gamma) = id$ for all loops γ based at p. Use parallel transport to show that E is a trivial vector bundle.

Construct a connection on the vector bundle $TS^1 \to S^1$ with nontrivial holonomy, and show this connection is flat. [So, trivial bundles with flat connections need not have trivial holonomy.]

(10) The Levi-Civita connection can be defined in a coordinate-free manner by the formula:

$$\begin{aligned}\langle \nabla_X Y, Z \rangle = \frac{1}{2}\Big(& X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ & - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle \Big),\end{aligned}$$

for all vector fields X, Y, Z, where the angle brackets denote the Riemannian inner product on tangent vectors. Using this formula, and the fact that the Christoffel symbols are defined by the equation $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$, derive the formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l}).$$

(11) Let ω be a closed p-form on a smooth Riemannian manifold (M, g) with the property that for any orthonormal vectors $e_1, \ldots, e_p$ at any point of M, we have $\omega(e_1, \ldots, e_p) \leq 1$. Suppose that S is a compact p-dimensional submanifold of M, with orientation $\mathcal{O}_S$, and the property that $\omega|_S$ is the Riemannian volume form on S. Prove that

$$\mathrm{Vol}(S) \leq \mathrm{Vol}(S')$$

for any oriented submanifold $(S', \mathcal{O}_{S'})$ of M for which the disjoint union of $(S, \mathcal{O}_S)$ and $(S', -\mathcal{O}_{S'})$ is the boundary of some smooth oriented submanifold $(T, \mathcal{O}_T)$ of M, with induced boundary orientation.