Differential Geometry Qualifying Exam
Spring 2017

Do any **6 problems**.

Note: Throughout this exam, all manifolds are $\mathcal{C}^\infty$ and connected, and all maps $\mathcal{C}^\infty$ are unless it is specifically stated otherwise.

1. Consider $S^n \subset \mathbb{R}^{n+1}$ with the induced metric and let $p = (0, 0, \ldots, 0, 1)$ be the north pole. Find an explicit expression of $\exp_p : T_pS^n \to S^n$. Prove that it has maximal rank except at the points $\vec{v} \in T_pS^n$ with $|\vec{v}| = \pm k\pi$, $k \in \mathbb{N}$, where it has rank 1.

2. Consider a mapping ϕ from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $\phi(x, y) = (xy, 1)$. Compute $\phi^*(dx)$, $\phi^*(dy)$, $\phi^*(ydx)$, and $\phi^*(xdy)$.

3. Suppose M is a complete Riemannian manifold and $V_1(t)$ and $V_2(t)$ are parallel vectors along a given curve $C(t)$.

   a) Prove that $g(V_1(t), V_2(t))$ does not depend on t.

   b) Prove that $g(V_1(t), C'(t))$ is constant when C is a geodesic.

   c) Now assume that M is $S^2 \subset R^3$, using (b) and the fact that great circles are geodesics, explicitly describe the parallel transport of a vector $(0, 5, 0)$ based at the north pole, $(0, 0, 1)$, along the great circle, $(0, \sin(t), \cos(t))$, to $(0, 1, 0)$ then along the equator, $(\sin(t), \cos(t), 0)$, to $(1, 0, 0)$ and back up the great circle $(\cos(t), \sin(t), 0)$ to the north pole.

4. Let G be a Lie group endowed with a bi-invariant metric. Prove that the map $\psi : G \to G$ given by $\psi(g) = g^{-1}$ is an isometry.

5. Let k, l be positive integers and $n = k + l$. Consider $G \subset Sl(n, \mathbb{R})$ consisting of those matrices $A = (a_{ij}) \in Sl(n, \mathbb{R})$ whose lower left-hand $k \times l$ block has all entries equal to zero, i.e. $A \in G$ if and only if $a_{ij} = 0$ for $i \leq k$, $j \leq l$. Prove that G is a Lie subgroup of $Sl(n, \mathbb{R})$ and describe the Lie algebra of G.

6. Let M be a Riemannian manifold and $p \in M$. For some normal ball $B_r(p) \subset M$, let $s_p : B_r(p) \to B_r(p)$ be the local diffeomorphism defined by

$$s_p(\exp_p(\vec{v})) = \exp_p(-\vec{v}).$$

   Suppose that s_p is an isometry from some geodesic ball $B_r(p)$. Prove that $(\nabla R)_p = 0$, where R is the Riemann curvature tensor.

7. Let $F : M \to N$ be a smooth mapping of differentiable manifolds. Assume that M is connected and $F_* \equiv 0$. Prove that F is constant i.e. that there exists $p \in N$ such that $F(q) = p$ for every $q \in M$.

8. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere centered at the origin. Show that if two maps f and g from a manifold M to $\mathbb{S}^n$ satisfy

$$|f(x) - g(x)| < 2$$

for all $x \in M$, then f is homotopic to g, the homotopy being smooth if f and g are smooth.

9. a) State the definition of the gradient of a function $f : M \longrightarrow \mathbb{R}$ on a Riemannian manifold M and the formula that relates the gradient, the exterior derivative and the Riemannian metric.

   b) Let $\gamma : (\alpha, \beta) \longrightarrow M$ be an integral curve of the gradient $\text{grad} f$ of a function f on M. Assume that $\text{grad} f$ is nonzero at all points of γ. Show that if $t \neq s$, $s, t \in (\alpha, \beta)$ then $\gamma(s) \neq \gamma(t)$. *Hint:* Compute $(f \circ \gamma)'$.

   c) Show that the vector field $V(x, y, z) = (-y, x, 0)$ is tangent to $S^2 \subset \mathbb{R}^3$. Is V considered as a vector field on S^3 the gradient of a function? (In this problem, we identify tangent vectors to S^2 or $\mathbb{R}^3$ with vectors in $\mathbb{R}^3$.)

10. Let M be a complete Riemannian manifold, $p \in M$, and let $\vec{v}, \vec{w} \in T_pM$. Let $\gamma(t) = \exp_p(\vec{v}t)$ be a geodesic. Prove that

$$J(t) = (d \exp_p)_{t\vec{v}}(t\vec{w})$$

is a Jacobi field along γ with $J(0) = 0$.

11. Let M be a differentiable manifold. Let $p : TM \to M$ be the natural projection that sends a tangent vector $v \in T_xM$ to x.

   a) Let $x^1, x^2, \ldots, x^n$ be local coordinates on an open subset $U \subset M$. Describe the cooordinate system on $p^{-1}(U)$ that can be derived from the local coordinates on U.

   b) Prove that, for every differentiable manifold M, TM is orientable.