Qualifying Exam Questions, Spring 2018
Differential Geometry

Do any **six** of the following problems.

Problem 1

For any integer $0 \leq k \leq n$, let $G_{k,n} = G_k(\mathbb{R}^n)$ denote the set of all k-dimensional linear subspaces of $\mathbb{R}^n$. Prove that $G_{k,n}$ can be given the structure of a smooth manifold of dimension $k(n-k)$.

Problem 2

Consider the map

$$f : S^2 \subset \mathbb{R}^3 \to \mathbb{R}^4, \qquad (x, y, z) \mapsto (x^2 - y^2, xy, yz, zx)$$

Show that this map induces an embedding from $\mathbb{R}\mathrm{P}^2$ to $\mathbb{R}^4$

Problem 3

Let M be a smooth manifold with tangent bundle TM. The *complexified tangent bundle* $TM \otimes \mathbb{C}$ is denoted $T^{\mathbb{C}}M$, and sections of $T^{\mathbb{C}}M$ are called *complexified vector fields.*

An *almost complex structure* on M is a bundle endomorphism J of TM satisfying $J^2 = -I$, where I is the identity endomorphism. Denote also by J its complex linear extension to an endomorphism of $T^{\mathbb{C}}M$.

(a) For each point $p \in M$, let $T_p^{1,0}M$ (resp. $T_p^{0,1}M$) be the $\sqrt{-1}$ (resp. $-\sqrt{-1}$) eigenspace of J in $T^{\mathbb{C}}M$. Show that these form smooth complex subbundles $T^{1,0}M$ and $T^{0,1}M$ of $T^{\mathbb{C}}M$. Show that the correspondence $X \mapsto X - \sqrt{-1}JX$ gives an isomorphism of real bundles $TM \sim T^{1,0}M$.

(b) Define a map N_J from smooth sections of $T^{1,0}M \otimes T^{1,0}M$ to sections of $T^{0,1}M$ by

$$N_J(V_1, V_2) = \frac{1}{2}(I + \sqrt{-1}J)[V_1, V_2],$$

where $[.,.]$ denotes the Lie bracket of complexified vector fields. Show that in fact N_J is a tensor.

(c) Show that if $\dim_{\mathbb{R}} M = 2$, then N_J vanishes identically.

Problem 4

(a) Give a conformally flat metric on $\mathbb{R}^N$ which has sectional curvature 1.

(b) Is it possible to find a metric on $\mathbb{R}^N$ that has sectional curvature -1?

(c) Is it possible to find a conformally flat metric on $\mathbb{R}^2$ which has sectional curvature -1?

Problem 5 (Möbius bundle)

Define an equivalence relation on $\mathbb{R}^2$ by declaring $(x,y) \sim (x',y')$ if and only if $(x',y') = (x+n, (-1)^n y)$ for some $n \in \mathbb{Z}$. Let $E = \mathbb{R}^2/\sim$ denote the quotient space, and $q : \mathbb{R}^2 \to E$ the quotient map.

Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first factor, and $\epsilon : \mathbb{R} \to S^1$ be the map $\epsilon(x) = e^{2\pi i x}$. The map $\epsilon \circ \pi_1$ is constant in each equivalence class, so it descends to a map $\pi : E \to S^1$.

(a) Show that E has a unique smooth structure such that q is a smooth covering map.

(b) Show that $\pi : E \to S^1$ is a smooth vector bundle.

(c) Show that it is not a trivial vector bundle.

Problem 6

Show that the only solutions on $\mathbb{R}^3$ of both $(\partial_x + x\partial_y)f = 0$ and $(\partial_y + y\partial_z)f = 0$ are constant.

Problem 7

Consider a regular smooth curve $f : \mathbb{R} \to \mathbb{R}^3$, which is such that its curvature κ and its torsion τ never vanish. If $f(t)$ belongs to a sphere of radius $R > 0$, find a relation that is satisfied by κ, τ.

Problem 8

(a) Prove that every smooth manifold admits a Riemannian metric.

(b) Prove that every smooth Riemannian manifold (M,g) admits a unique torsion free connection which is compatible with the metric.

Problem 9

(a) Give explicitly the Riemann curvature tensor and Ricci curvature of the round sphere S^n.

(b) What is the sectional curvature of $S^2 \times S^2$ (endowed with the product metric)?

Problem 10

(a) State and prove the Hopf-Rinow Theorem.

(b) Let (M,g) be a Riemannian manifold satisfying the following property: given any points $p,q \in M$, there exists an isometry f of M such that $f(p) = q$. Show that M is geodesically complete.

Problem 11

Let (M,g) be a Riemannian manifold.

(a) Let $\gamma : [0,1] \to M$ be a smooth minimizing curve. Show that γ is a geodesic.

(b) Find the geodesics of the Poincaré upper half plane.