Do any 6 problems. Note: Throughout this exam, all manifolds are $C^\infty$ and connected, and all maps are $C^\infty$ unless it is specifically stated otherwise.

1. Let $G$ be the set of $2 \times 2$ invertible matrices with real entries and $0$ in the lower left corner. Show that $G$ is a smooth manifold, is not connected and can be parametrized by a single chart. Show that $G$ is homotopy-equivalent to a space consisting of four points and that every closed differential form on $G$ of positive degree is exact.

2. Let $N \subset M$ be a closed, embedded submanifold of a smooth manifold $M$. Show that every Riemannian metric $g$ on $N$ is the restriction to $N$ of a Riemannian metric $h$ on $M$.

3. Let $P$ be a geodesic polygon in the hyperbolic plane of constant curvature $-1$ with $n$ sides and interior angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. Prove that the area of $P$ is equal to $(n-2)\pi - \sum_{i=1}^{n} \alpha_i$.

4. Let $S$ be a surface of revolution in $\mathbb{R}^3$ equipped with the induced metric. Show that the curves of intersection of $S$ with planes passing through the axis of revolution are geodesics on $S$. Hint: No calculations are necessary to do this problem.

5. Let $M$ be a Riemannian manifold and $\gamma : [0, a] \rightarrow M$ a smooth curve parametrized by arc length. Let $V$ be a smooth vector field along $\gamma$ vanishing at the endpoints, $V(0) = 0$, $V(a) = 0$.
   a) Prove that for some $\epsilon > 0$ there exists a parametrized surface $\alpha : [0, a] \times (-\epsilon, \epsilon) \rightarrow M$ which is a proper variation of $\gamma$, i.e. $\alpha(t, 0) = \gamma(t)$ for all $t \in [0, a]$ and $\alpha(0, s) = \gamma(0)$ and $\alpha(a, s) = \gamma(a)$ for all $s \in (-\epsilon, \epsilon)$ such that $\partial \alpha / \partial s(t, 0) = V(t)$ for all $t \in [0, a]$.
   b) If $\gamma_s(\cdot) = \alpha(\cdot, s)$ prove the first variation of arc length formula
   $$ \frac{d}{ds}|_{s=0} L(\gamma_s) = - \int_0^a \left< V, \frac{D}{dt} \gamma' \right> dt. $$
   c) Conclude that if $\gamma$ minimizes the length between $p = \gamma(0)$ and $q = \gamma(a)$ among all smooth curves from $p$ to $q$, then $\gamma$ is a geodesic.

6. Show that if a Riemannian manifold $M$ has the property that for every two points $p, q \in M$ and orthonormal pairs of tangent vectors $v, w \in T_p M$ and $\tilde{v}, \tilde{w} \in T_q M$ there exists a local isometry taking $p$ to $q$, $v$ to $\tilde{v}$, and $w$ to $\tilde{w}$, then $M$ has a constant sectional curvature.
7. Let \( S \) be a surface equipped with a complete Riemannian metric and \( \gamma : (-1, 1) \to S \) a smooth curve. Suppose \( X_t \) is a parallel vector field along \( \gamma \) such that \(|X_t| \equiv 1\) and \( \langle X_t, \dot{\gamma}(t) \rangle \equiv 0 \). Consider the mapping \( E : (-1, 1) \times (-\infty, \infty) \to S \) given by

\[
E(s, t) = \exp_{\gamma(s)}(tX_s).
\]

Show that the curves \( s \to E(s, t_0) \) are perpendicular to geodesics \( t \to E(s_0, t) \) for all \( (s_0, t_0) \).

8. Consider the differential form \( \omega = xydz - ydxz + zdx \wedge dy \) in \( \mathbb{R}^3 \). Prove that \( \omega \) is invariant under rotations, i.e. if \( A \) is an orthogonal transformation of \( \mathbb{R}^3 \) then \( A^* \omega = \omega \), and that

\[
\int_{S^2} \omega > 0.
\]

9. Consider the mapping \( \tilde{F} : S^2 \to \mathbb{R}^3 \) given by the formula

\[
\tilde{F}(x, y, z) = (yz, xz, xy).
\]

a) Show that \( \tilde{F} \) induces a map \( F \) from the real projective plane \( \mathbb{R}P^2 \) to \( \mathbb{R}^3 \).

b) Show that \( F \) is an immersion but is not injective.

c) Let \( \pi : (u, v, w) \to (u, v) \). At which points of the projective plane does \( \pi \circ F \) fail to be an immersion.

10. Let \( \omega \) be a 1–form on \( S^2 \) invariant under all orthogonal transformations of \( \mathbb{R}^3 \). Show that \( \omega \) must vanish identically.