# Differential Geometry
## Spring 2019

Do any 6 problems. *Note:* Throughout this exam, all manifolds are $\mathcal{C}^\infty$ and connected, and all maps are $\mathcal{C}^\infty$ unless it is specifically stated otherwise.

1. Let $G$ be the set of 2 by 2 invertible matrices with real entries and 0 in the lower left corner. Show that $G$ is a smooth manifold, is not connected and can be parametrized by a *single* chart. Show that $G$ is homotopy-equivalent to a space consisting of four points and that every closed differential form on $G$ of positive degree is exact.

2. Let $N \subset M$ be a closed, embedded submanifold of a smooth manifold $M$. Show that every Riemannian metric $g$ on $N$ is the restriction to $N$ of a Riemannian metric $h$ on $M$.

3. Let $P$ be a geodesic polygon in the hyperbolic plane of constant curvature $-1$ with $n$ sides and interior angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. Prove that the area of $P$ is equal to $(n-2)\pi - \sum_{i=1}^n \alpha_i$.

4. Let $S$ be a surface of revolution in $\mathbb{R}^3$ equipped with the induced metric. Show that the curves of intersection of $S$ with planes passing through the axis of revolution are geodesics on $S$. *Hint:* No calculations are necessary to do this problem.

5. Let $M$ be a Riemannian manifold and $\gamma : [0, a] \longrightarrow M$ a smooth curve parametrized by arc length. Let $V$ be a smooth vector field along $\gamma$ vanishing at the endpoints, $V(0) = 0$, $V(a) = 0$.

   a) Prove that for some $\epsilon > 0$ there exists a parametrized surface $\alpha : [0, a] \times (-\epsilon, \epsilon) \longrightarrow M$ which is a proper variation of $\gamma$, i.e. $\alpha(t, 0) = \gamma(t)$ for all $t \in [0, a]$ and $\alpha(0, s) = \gamma(0)$ and $\alpha(a, s) = \gamma(a)$ for all $s \in (-\epsilon, \epsilon)$ such that $\partial\alpha/\partial s(t, 0) = V(t)$ for all $t \in [0, a]$.

   b) If $\gamma_s(\cdot) = \alpha(\cdot, s)$ prove the first variation of arc length formula
   $$\frac{d}{ds}_{|s=0} L(\gamma_s) = -\int_0^a < V, \frac{D}{dt}\gamma' > \, dt.$$

   c) Conclude that if $\gamma$ minimizes the length between $p = \gamma(0)$ and $q = \gamma(a)$ among all smooth curves from $p$ to $q$, then $\gamma$ is a geodesic.

6. Show that if a Riemannian manifold M has the property that for every two points $p, q \in M$ and orthonormal pairs of tangent vectors $v, w \in T_pM$ and $\tilde{v}, \tilde{w} \in T_qM$ there exists a local isometry taking $p$ to $q$, $v$ to $\tilde{v}$, and $w$ to $\tilde{w}$, then $M$ has a constant sectional curvature.

7. Let $S$ be a surface equipped with a complete Riemannian metric and $\gamma : (-1, 1) \to S$ a smooth curve. Suppose $X_t$ is a parallel vector field along $\gamma$ such that $|X_t| \equiv 1$ and $\langle X_t, \dot{\gamma}(t) \rangle \equiv 0$. Consider the mapping $E : (-1, 1) \times (-\infty, \infty) \to S$ given by

$$E(s, t) = \exp_{\gamma(s)}(tX_s).$$

Show that the curves $s \to E(s, t_0)$ are perpendicular to geodesics $t \to E(s_0, t)$ for all $(s_0, t_0)$.

8. Consider the differential form $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$ in $\mathbb{R}^3$. Prove that $\omega$ is invariant under rotations, i.e. if $A$ is an orthogonal transformation of $\mathbb{R}^3$ then $A^*\omega = \omega$, and that

$$\int_{S^2} \omega > 0.$$

9. Consider the mapping $\tilde{F} : S^2 \to \mathbb{R}^3$ given by the formula

$$\tilde{F}(x, y, z) = (yz, xz, xy).$$

   a) Show that $\tilde{F}$ induces a map $F$ from the real projective plane $\mathbb{R}P^2$ to $\mathbb{R}^3$.
   b) Show that $F$ is an immersion but is not injective.
   c) Let $\pi : (u, v, w) \to (u, v)$. At which points of the projective plane does $\pi \circ F$ fail to be an immersion.

10. Let $\omega$ be a 1–form on $S^2$ invariant under all orthogonal transformations of $\mathbb{R}^3$. show that $\omega$ must vanish identically.