Logic Comprehensive Exam
Autumn 2015

Part Zero

Answer both of the following questions.

1. Formulate precisely the Completeness Theorem and the Compactness Theorem, and outline the proof of one of them.

2. Formulate precisely Gödel's First and Second Incompleteness Theorems, and outline the proof of one of them.

Part One

Do four of the following eight problems.

1. Let $\mathcal{L}$ be a first order language and let $\mathfrak{M}$ be a finite $\mathcal{L}$-structure. Show that $\mathfrak{M}$ is an existentially closed model of $Th(\mathfrak{M})$.

2. Let $\mathcal{L} = \{<, P\}$ where P is a unary predicate and $<$ is a binary predicate. Consider the theory T axiomatized by:

   - $\forall x(x \not< x)$
   - $\forall x \forall y(x < y \vee y < x \vee x = y)$
   - $\forall x \forall y(x < y \rightarrow (y \not< x \wedge x \neq y))$
   - $\forall x \forall y \forall z((x < y \wedge y < z) \rightarrow x < z)$
   - $\forall x \forall y(x < y \rightarrow \exists z(x < z \wedge z < y \wedge Pz))$
   - $\forall x \forall y(x < y \rightarrow \exists z(x < z \wedge z < y \wedge \neg Pz))$
   - $\forall x \exists y(x < y)$
   - $\forall x \exists y(y < x)$

   Show that T is complete.

3. Let $\mathcal{L} = \{P_n : n \in \mathbb{N}\}$ where the P_n are unary predicates. Consider the theory T axiomatized by:

   - $\exists x_1 \ldots \exists x_n(\bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j \wedge \bigwedge_{1 \leq i \leq n} P_m x_i)$ for all $m, n \in \mathbb{N}$.
   - $\neg \exists x(P_i x \wedge P_j x)$ for all $i \neq j \in \mathbb{N}$.

   Describe all of the countable models of T. Which if any of these is saturated? Which if any is prime? Justify your answers.

4. Let $\mathcal{L}$ be a first order language and let $\mathfrak{M}_i$ be finite $\mathcal{L}$-structures for $i \in I$ where I is any index set. Let $\mathcal{U}$ be an ultrafilter on I and let $\mathfrak{M} = (\Pi_{i \in I} \mathfrak{M}_i)/\mathcal{U}$ be the ultraproduct. Show there is no $\mathcal{L}$-formula $\psi(x, y)$ so that $\psi(M^2)$ is a linear ordering on M with no largest element.

5. Let T be a complete and model complete theory which is axiomatizable by a set of universal $\mathcal{L}$-sentences. Suppose that $\mathfrak{M} \models T$ and that $f : M \to M$ is a $\emptyset$-definable function (i.e. the graph of f is an $\emptyset$-definable subset of M^2). Show that if $a \in M$ then there is an $\mathcal{L}$-term $t(x)$ so that $f(a) = t(a)$.

6. Let T be a complete theory in a countable language. Suppose that $\mathfrak{M}$ is a countable model of T which is atomic and countably universal (i.e. if $\mathfrak{N}$ is any other countable model of T then $\mathfrak{N}$ elementarily embeds into $\mathfrak{M}$). Show that $\mathfrak{M}$ is saturated.

7. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $\varphi(x)$ be an $\mathcal{L}$-formula. Show that $\varphi(M)$ is finite if and only if for all $\mathcal{L}$-structures $\mathfrak{N}$ if $\mathfrak{M} \preceq \mathfrak{N}$ then $\varphi(M) = \varphi(N)$.

8. Let $\mathcal{L}$ be a countable language which includes a countable set of constant symbols $\{c_i : i \in \mathbb{N}\}$. Let T be a complete $\mathcal{L}$-theory so that for any $\mathcal{L}$-formula $\varphi(x)$ the sentence $\varphi(c_i) \in T$ for some $i \in \mathbb{N}$. Show that there is a model $\mathfrak{N}$ of T so that $N = \{c_i^{\mathfrak{N}} : i \in \mathbb{N}\}$.

Part Two

Do four of the following eight problems.

1. Let $\mathcal{L}$ be a finite signature with no function symbols. Suppose that T is a complete $\mathcal{L}$-theory with quantifier elimination. Show that the Boolean algebra of all $\mathcal{L}$-sentences, modulo equivalence under T, is finite.

2. Let T be a complete theory which is axiomatized by a decidable set A. Is it possible that T admits elimination of quantifiers, but does not admit effective elimination of quantifiers? Either give an example of such a T (and sketch a proof that it is an example of such a theory), or show that no such T exists.

3. Let A_0 be an arbitrary finite subset of the theory of the structure $\mathcal{N} = (\omega, 0, S, <, +, \cdot, E)$. (Here E represents exponentiation, as a binary function.) Determine how many complete consistent extensions of A_0 exist in this signature, and prove your answer.

4. Let S be an arbitrary c.e. subset of ω. Prove that there exists a 1-reduction from its jump

$$S' = \{e \in \omega \ : \ \Phi_e^S(e) \downarrow\}$$

to the set

$$\mathbf{Fin} = \{e \ : \ W_e \text{ is finite}\}.$$

(It is not difficult to describe an actual 1-reduction, but other proofs are also possible.)

5. Let U and V be Π_1 subsets of ω, with $\overline{U} \subseteq V$. Prove that the intersection $(U \cap V)$ is a least upper bound of the sets U and V under Turing reducibility.

6. Let f be a total computable function with $\operatorname{ran}(f) \subseteq \{0, 1, \ldots, 9\}$, and write $a_n = f(n)$. One might think of f as giving a "computable real number" with decimal representation $0.a_0a_1a_2a_3\ldots$. Set $S_1(f)$ to be the set of those finite strings ρ of digits which "appear" in this number, i.e., such that for some m, we have $(\forall i < |\rho|)\ a_{m+i} = \rho(i)$.

   - Show that $S_1(f)$ must be computably enumerable.
   - Give an example of a computable function f for which $S_1(f)$ is not computable.

- Now define $S_2(f)$ to be the set of those finite strings ρ which appear at least twice in this number. (That is, some two distinct values m and m' both have the property described above for this ρ.) Prove that $S_2(f) \leq_T S_1(f)$.

7. The *Finite Union Axiom* (which is not normally included in the axiom set **ZFC**) states that, for every two sets a and b, the union $a \cup b$ is also a set. State this formally in the language of set theory. Then show that, given the Power Set, Extensionality, and Comprehension Axioms, the Finite Union Axiom implies the Pairing Axiom.

   Conclude (with an explanation) that in **ZFC**, every proof of the Finite Union Axiom from the Union Axiom requires use of the Pairing Axiom. Mention any assumptions you make about the consistency, inconsistency, or independence of various sets of axioms.

8. Prove that for every set B of ordinals, the union $\bigcup B$ is also an ordinal.