Logic Comprehensive Exam
Summer 2018

Three Parts

Part Zero

1. State and prove the Tarski-Vaught criterion for elementary substructures.
2. State and prove Cantor's theorem on countable endless dense linear orders.
3. State the compactness theorem of first-order logic and sketch a proof of it.
4. State two equivalent formulations of the axiom of choice, and sketch a proof of their equivalence.

Part One

Do four of the following eight problems.

1. Let $\mathcal{L}$ be a first-order language and $\mathfrak{M}$ be an $\mathcal{L}$-structure. Let $\mathcal{L}(M)$ be $\mathcal{L}$ expanded with constants for the individual elements of $\mathfrak{M}$ and let $\mathfrak{M}_M$ be the natural expansion of $\mathfrak{M}$ to and $\mathcal{L}(M)$-structure. Show that for any $\mathcal{L}$ structure $\mathfrak{N}$ there is an elementary embedding $f : \mathfrak{M} \to \mathfrak{N}$ if and only if there is an expansion of $\mathfrak{N}$ to a model of $\mathrm{Th}(\mathfrak{M}_M)$.

2. Let $\mathcal{L}$ be the language with a single binary relation R. Let T be the theory axiomatized by stating that R is an equivalence relation with exactly two infinite classes and no other classes. Show that T has quantifier elimination.

3. Let $\mathcal{L}$ be the first-order language consisting of countably many unary predicates U_i for $i \in \omega$. Let T be the $\mathcal{L}$-theory axiomatized by:

   - $\forall x \neg(U_i x \wedge U_j x)$ for all $i \neq j \in \omega$;
   - $\exists x_1, \ldots, x_n(\bigwedge_{1 \leq i \leq n} U_j x_i \wedge \bigwedge_{1 \leq i \neq k \leq n} x_i \neq x_k)$ for all $j, n \in \omega$.

   Describe all of the countable models of T and indicate which, if any, are prime or saturated. Justify you answer.

4. Let $\mathfrak{M}_i$ for $i \in \omega$ be finite $\mathcal{L}$-structures. Let $\mathcal{U}$ be an ultrafilter on ω. Show that in the ultraproduct $\mathfrak{M} = (\Pi_{i \in \omega} \mathfrak{M}_i)/\mathcal{U}$ there is no formula $\varphi(x, y)$ which defines an infinite linear order with no largest element.

5. Let $\mathcal{L}$ be a first order language consisting of a binary relation symbol $<$ and a unary predicate P. Let $\mathfrak{M}$ be the $\mathcal{L}$-structure with universe the real numbers where $<$ is interpreted as the usual order on the real numbers and P is interpreted as the set of rational numbers. Show that $\mathrm{Th}(\mathfrak{M})$ is ω-categorical.

6. Let $\mathcal{L}$ be a countable first-order language. Show that a countable $\mathcal{L}$-structure $\mathfrak{M}$ is countably saturated if and only if it is countably universal and homogeneous.

7. Let T be a theory in a countable language $\mathcal{L}$. Let $\varphi_i(x)$ for $i \in \omega$ be a collection of formulas in the free variable x and let $\psi(x)$ be another such formula. Suppose that for any other formula $\sigma(x)$ if $T \models \forall x(\sigma(x) \to \psi(x))$ then $T \models \exists x(\sigma(x) \wedge \varphi_i(x))$ for some $i \in \omega$. Show that there is a model $\mathfrak{M}$ of T so that $\psi(\mathfrak{M}) \subseteq \bigcup_{i \in \omega} \varphi_i(\mathfrak{M})$

8. A theory T without finite models is called *strongly minimal* if in any model $\mathfrak{M}$ of T any definable subset $X \subseteq M$ is either finite or co-finite. (More specifically if $\mathfrak{M} \models T$, $\varphi(x, \overline{y})$ is a formula, and $\overline{a} \in M^{|\overline{y}|}$ then the set of realizations of $\varphi(x, \overline{a})$ in $\mathfrak{M}$ is either finite or co-finite in M.) Suppose that T is strongly minimal and show that if $\varphi(x, \overline{y})$ is any formula then there is a natural number N so that if $\mathfrak{M} \models T$ and $\overline{a} \in M^{|\overline{y}|}$ then if $|\{b \in M : \mathfrak{M} \models \varphi(b, \overline{a})\}| > N$ then $\{b \in M : \mathfrak{M} \models \varphi(b, \overline{a})\}$ is infinite.

Part Two

Do four of the following eight problems.

1. Let T be a recursive, consistent set of sentences containing the axioms of ZFC. A sentence φ is *independent* of T if neither φ nor $\neg\varphi$ are provable from T. Show that the set of sentences that are independent of T is not recursively enumerable.

2. Show that a language is decidable iff it has an enumerator with the property that if it outputs a word v before a word w, then the length of v is less than or equal to the length of w.

3. Let $\{\ulcorner M_1 \urcorner, \ulcorner M_2 \urcorner, \ldots\}$ be a recursively enumerable language consisting of descriptions of deciders M_1, M_2, ... Suppose the underlying alphabet consists of the symbols 0 and 1. Show that there is a decider whose language is different from the languages decided by M_1, M_2, ...

4. Show in ZF, assuming the consistency of Peano arithmetic, that there is a bijection between $\mathcal{P}(\omega)$ and the set of complete and consistent theories (in the language of arithmetic) extending Peano Arithmetic.

5. Let κ be an uncountable regular cardinal, and suppose that $\langle X_\alpha \mid \alpha < \theta \rangle$ is a strictly increasing sequence (of length θ) of subsets of κ, such that $\bigcup_{\alpha<\theta} X_\alpha = \kappa$. Show:

   (a) $\theta < \kappa^+$.

   (b) for every nonzero $\gamma < \kappa^+$, there is such a sequence of length γ.

   (c) Assuming that for each $\alpha < \theta$, the cardinality of X_α is less than κ, it follows that $\theta = \kappa$.

6. The principle of Dependent Choices (DC) says that whenever $\langle u, r \rangle$ is a binary system with the property that for every $y \in u$, there is an $x \in u$ such that xry, it follows that there is an infinite decreasing chain in $\langle u, r \rangle$, that is, a function $f : \omega \longrightarrow u$ such that for all $n \in \omega$, $f(n+1) r f(n)$. The countable Löwenheim-Skolem Theorem (LS_ω) says that every infinite model M of a countable first order language language has a countable elementary submodel. Show in ZF that LS_ω is equivalent to DC.

7. Let κ be a strongly inaccessible cardinal, and let $M = \langle V_\kappa, \in \restriction V_\kappa \rangle$, a model of the language of set theory. Let $N = \langle X, \in \restriction X \rangle \prec M$ be a countable elementary submodel.

   (a) Show that $N \cap \omega_1 \in \omega_1$ and $\omega_1 \in X$.

   (b) Show that X is not transitive.

   (c) Show that N is extensional and well-founded.

   (d) Let $f : N \longrightarrow \langle U, \in \restriction U \rangle$ be the Mostowski collapse and isomorphism. Show that $f \restriction \omega_1$ is the identity and $f(\omega_1) = \omega_1 \cap X$.

8. Prove the following statements (in ZFC):

   (a) Let u be a transitive set. Whenever $\varphi(\vec{x})$ is a Δ_0-formula and $\vec{a} \in u$, then
   $$\varphi(\vec{a}) \text{ holds iff } \langle u, \in \restriction u \rangle \models \varphi(\vec{a}).$$

   (b) Let $\psi(\vec{x})$ be a Σ_1-formula, let κ be an infinite cardinal, let $\vec{a} \in H_\kappa$ (where $H_\kappa = \{b \mid \mathrm{card}(b) < \kappa\}$). Then
   $$\psi(\vec{a}) \text{ holds iff } \langle H_\kappa, \in \restriction H_\kappa \rangle \models \psi(\vec{a}).$$