Logic Qualifying Exam
Three Parts
August 2020

Part Zero

Provide an attribution/name and a sketch of a proof for any **THREE** of the following results:

1. If a substructure $\mathcal{A}$ of a structure $\mathcal{B}$ is closed under Skolem functions, then $\mathcal{A} \prec \mathcal{B}$.

2. Every finite order-preserving partial function from one countable dense linear order without endpoints to another can be extended to an isomorphism.

3. If every finite subset of a collection of sentences in a fixed first order language has a model, then the whole collection has a model.

4. Let $(\mathcal{A}_i \mid i < \lambda)$ be a sequence of structures of some fixed first order language $\mathcal{L}$ such that for $i < j < \lambda$, $\mathcal{A}_i \prec \mathcal{A}_j$. Let $\mathcal{A}_\lambda$ be the $\mathcal{L}$-structure whose universe is the union of the universes of the structures $\mathcal{A}_i$, for $i < \lambda$, and whose interpretation of the symbols in the language is given by $\dot{c}^{\mathcal{A}_\lambda} = \dot{c}^{\mathcal{A}_i}$ for any/all $i < \lambda$ if $\dot{c}$ is a constant symbol, $\dot{R}^{\mathcal{A}_\lambda} = \bigcup_{i<\lambda} \dot{R}^{\mathcal{A}_i}$ if $\dot{R}$ is a relation symbol, and $\dot{f}^{\mathcal{A}_\lambda}(a_0, \ldots, a_{n-1}) = b$ if $\dot{f}^{\mathcal{A}_i}(a_0, \ldots, a_{n-1}) = b$ for any/all $i < \lambda$ such that $a_0, \ldots, a_{n-1}$ belong to the universe of $\mathcal{A}_i$, when $\dot{f}$ is an n-ary function symbol. Then for every $i < \lambda$, $\mathcal{A}_i \prec \mathcal{A}_\lambda$.

5. Every infinite tree in which every node has at most finitely many immediate successors has an infinite branch.

6. If a and b are sets and there are an injection from a into b and an injection from b into a, then there is a bijection between a and b.

Part One

Do **THREE** of the following six problems. All syntax below is assumed to be first order, with equality as a logical symbol (i.e. always part of any of the languages considered).

1. Assume that a sentence φ and a theory T have exactly the same models. Prove that there is a finite subset $S \subseteq T$ such that φ and S have exactly the same models.

2. Prove that the ordering of the rationals is an elementary substructure of the ordering of the reals. [You may skip the calculational details if you explain the principle well enough.]

3. Suppose L is a language whose single non-logical symbol is a binary relation symbol E. Consider the L-theory T of all L-structures in which E defines an equivalence relation with infinitely many E-classes, such that every E-class is infinite.

   (a) Write down an L-axiomatization of T.

   (b) Describe the countable models of T.

   (c) How many are there (up to isomorphism)?

   (d) Conclude what you can from this about the completeness of T.

4. Let T be the theory of vector spaces over $\mathbb{Q}$—in a language that contains a constant 0, a binary function symbol $+$, both with the natural interpretation, and for each $q \in \mathbb{Q}$ a unary function symbol f_q representing scalar multiplication by q.

   (a) Show that T is a complete theory

   (b) with quantifier elimination.

   (c) Is T ω-categorical? [Justify your answer.]

5. Let L be the language with a single non-logical symbol, a binary relation E. Let T be the theory stating that E is an equivalence relation with exactly three classes all of which are infinite. Show that T has quantifier elimination.

6. Let M_i, $i \in \omega$, be finite L-structures. Let U be an ultrafilter on ω. Show that in the ultraproduct $M = (\Pi_{i\in\omega} M_i)/U$ there is no formula $\varphi(x, y)$ which defines an infinite linear order with no largest element.

Part Two

Do THREE of the following six problems.

1. Prove that every complete theory with a computably enumerable axiom set is decidable.

2. Prove true or prove false: there is a universal total computable function, that is, a total computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that every total computable function occurs as f_n for some n, where f_n is the function defined by $f_n(m) = f(n, m)$.

3. Suppose that φ is a partial computable function whose domain is not computable. Prove that there exists an $n \in \omega$ such that in $\mathbb{N}$, φ does not halt on input n, but such that there exists a model of $\mathrm{A_E}$ in which φ does halt on input n.

4. Let TA (for "true arithmetic") be the set of Gödel codes of sentences true in the standard model of arithmetic $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$. Show that if $A \subseteq \mathbb{N}$ is definable in $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$, then $A <_T TA$. (Notation: $X <_T Y$ means that X is Turing computable from an oracle for Y, but not conversely. Be sure to show the strictness, as well as the existence of a reduction.)

5. Show that, under the axiom system **ZF**, the following two versions of the Axiom of Choice are equivalent.

   (AC): For each set R of ordered pairs, there is a function $H \subseteq R$ with $\mathrm{dom}(H) = \mathrm{dom}(R)$.

   (WO): Every set has a well-ordering.

6. Let $(A, <)$ and $(B, \prec)$ be well-orders of two sets A and B. Prove that if these two orders are isomorphic, then the isomorphism between them is unique.