Logic Qualifier
Spring 2013

Part Zero

Choose four of the following six theorems. Give a precise formulation, and sketch the proof. Not all details must be included.

(1) The Tarski-Vaught Test for elementary substructures.
(2) The (downward) Löwenheim-Skolem Theorem.
(3) Either the Finiteness Theorem (a.k.a., Compactness Theorem), or the Completeness Theorem.
(4) Either the First or the Second Incompleteness Theorem.
(5) Tarski's Undefinability of Truth Theorem.
(6) The non-computability of the Halting Problem.

Part One

Do four of the following eight problems.

(1) Using Tarski-Seidenberg—that is to say, quantifier elimination for the real ordered field in the signature $\langle 0, 1; +, -, \cdot; < \rangle$—show that the field of real numbers viewed as a pure field—that is to say, in the signature $\langle 0, 1; +, -, \cdot \rangle$—is model-complete.

(2) Let L be a language and T an L-theory. Let σ_n and τ_n, for $n \in \mathbb{N}$, be sentences in this language. Show that if the infinitary conjunction $\bigwedge_{n\in\mathbb{N}} \sigma_n$ is equivalent modulo T to the infinitary disjunction $\bigvee_{n\in\mathbb{N}} \tau_n$, then these two infinitary sentences are actually equivalent modulo T to a (first-order) sentence, as follows. For each n, let T_n be the theory consisting of T, all σ_i for $i \in \mathbb{N}$, and $\neg\tau_1, \ldots, \neg\tau_n$. Show that some T_n must be inconsistent, by using an ultraproduct construction or appealing to the Compactness/Finiteness Theorem. Now deduce that the infinitary sentence $\bigvee_{n\in\mathbb{N}} \tau_n$ is first-order modulo T.

(3) Prove that if $(\mathcal{M}_i | i \in \mathbb{N})$ is an elementary chain of L-structures, in the sense that $\mathcal{M}_i \preceq \mathcal{M}_{i+1}$ for each i, and if $\mathcal{M}$ is the union of this chain, then $\mathcal{M}_0 \preceq \mathcal{M}$.

(4) Let $\mathbf{S}_L$ the Stone space of L consisting of all complete L-theories T. Show that a theory T is finitely axiomatizable if and only if it is an isolated point of $\mathbf{S}_L$. (Recall that a point x in a topological space S is called *isolated* if the singleton $\{x\}$ is open.)

(5) Give two non-isomorphic algebraically closed fields that have some nontrivial ultrapowers which are isomorphic. Substantiate all your claims about these fields.

(6) Express the following statements in first order logic.
   (a) In the empty language, for a positive integer p: "there are at most p different elements."
   (b) In the language of rings $\{+, \cdot, 0, 1\}$: "the characteristic is p."
   (c) In the language of graphs $\{R\}$: "there is a path of length exactly 3 (i.e. a path with exactly 3 edges) between x and y."
   (d) In the language of orderings $\{<\}$: "every element has an immediate predecessor."

(7) Let S and T be L-theories such that $S_\forall \subseteq T \subseteq S$. Show that if T admits elimination of quantifiers, then $\mathrm{Mod}(T) = \mathrm{Mod}(S)$.

(8) Let $\mathbb{R}_\natural$ be a nontrivial ultrapower of the reals $\mathbb{R}$ and view $\mathbb{R}$ as a subfield via the diagonal embedding. Call a non-zero element $\alpha \in \mathbb{R}_\natural$ an *infinitesimal*, if $0 < |\alpha| < r$, for every positive $r \in \mathbb{R}$; call r *infinite* if $1/\alpha$ is infinitesimal; and call α *finite* if it is not infinite (or zero). Give an example of an infinitesimal. Show that the finite elements form a ring R and the infinitesimals $\mathfrak{M}$ are an ideal in R. Show that for any element $\alpha \in \mathbb{R}_\natural$, either α is infinite or else there is a unique real number $r \in \mathbb{R}$, called the *standard part of* α, such that $\alpha - r$

Part Two

Do four of the following eight problems.

(1) Let $M \models \mathrm{PA}$ be nonstandard. Suppose that $\varphi(x)$ is a formula in the language of arithmetic $\{+, \times, <, 0, 1\}$ such that for every standard $n \in \mathbb{N}$, we have $M \models \varphi(n)$. Prove that there is a nonstandard $c \in M$ such that $M \models \forall x < c\, \varphi(x)$.

(2) Show that PA proves all Σ_1-sentences that are true in $\mathbb{N}$. Is there a recursive extension of PA that proves all Π_1-sentences that are true in $\mathbb{N}$?

(3) Suppose that $f(x, y)$ is a computable function. Use the Recursion Theorem to show that there is a natural number n such that $\varphi_n(y) = f(n, y)$.

(4) Prove that there is no computable procedure that decides whether a sentence φ holds in all structures of the language of arithmetic.

(5) Let $H = \{\langle e, n\rangle \mid \varphi_e(n)\downarrow\}$ and $K = \{e \mid \varphi_e(e)\downarrow\}$. Prove that $K \equiv_T H$. (The notation $X \equiv_T Y$ means that X is Turing computable with an oracle for Y and conversely.)

(6) Sketch the proof that the class of recursive functions is the same as the class of Turing computable functions.

(7) A set $A \subseteq \mathbb{N}$ is said to be *represented* in a theory T of the language of arithmetic if there is a formula $\varphi(x)$ such that for all $n \in \mathbb{N}$, we have

(a) $n \in A \to \mathrm{PA} \vdash \varphi(\underline{n})$ (where $\underline{n}$ is the term $\underbrace{1 + \cdots + 1}_{n}$),

(b) $n \notin A \to \mathrm{PA} \vdash \neg\varphi(\underline{n})$.

Show that the sets represented in PA are precisely the computable sets.

(8) Use Tarski's Undefinability of Truth Theorem to prove the First Incompleteness Theorem.