LOGIC QUALIFYING EXAM
SPRING 2019

Part 0

For any **FOUR** of the following results, give the statement and provide a proof.

(1) Tarski-Vaught test for elementary extensions.

(2) Elemenary Chain lemma.

(3) Downward Löwenheim-Skolem theorem.

(4) Upward Löwenheim-Skolem theorem.

(5) Łoś's lemma for ultraproducts.

(6) König's lemma for trees.

PART 1

Do any **FOUR** of the following problems.

(1) Prove that every set of rational numbers that is parametrically definable in $(\mathbb{Q}, <)$ is the union of finitely many open intervals and points.

(2) Prove that the only sets of rational numbers that are definable without parameters in $(\mathbb{Q},+)$ are $\varnothing$, $\{0\}$, $\mathbb{Q}$, and $\mathbb{Q} \setminus \{0\}$.

(3) Prove that every collection of subsets of a set I with the finite intersection property extends to an ultrafilter on I.

(4) Assume that a sentence ϕ and a theory T have exactly the same models. Prove that there is a finite subset $S \subseteq T$ such that ϕ and S have exactly the same models.

(5) Show that there is no first order sentence ϕ such that for every ordered set $(A, <)$, $(A, <) \models \phi$ if and only if $<$ well-orders A.

(6) Quote the results from model theory and algebra from which it follows that the theory of algebraically closed fields of characteristic 0 is $\aleph_1$-categorical, but is not $\aleph_0$-categorical.

(7) Describe all complete 1-types and all complete 2-types realized in the ordered set $(\mathbb{Z}, <)$.

(8) Prove that the theory of the group $(\mathbb{Z}, +)$ is not $\aleph_0$-categorical.

(9) Prove that the theory of the standard model of arithmetic $(\mathbb{N}, +, \times)$ does not have countable $\aleph_0$-saturated models.

(10) Let M be an infinite $\aleph_0$-saturated model. Prove that there are a and b in the domain of M such that $a \neq b$ and (M, a) and (M, b) are elementarily equivalent.

Part 2

Do **FOUR** of the following problems.

(1) Let $\pi : \omega \longrightarrow V_\omega$ be Ackermann's isomorphism. Show that there is no recursive set $R \subseteq \omega$ such that $\{\pi^{-1}(\sigma) \mid A_E \vdash \sigma\} \subseteq R$ and $\{\pi^{-1}(\sigma) \mid A_E \vdash \neg\sigma\} \subseteq \omega \setminus R$.

(2) How many consistent, complete extensions of A_E are there? Prove that your answer is correct.

(3) Assuming that ZFC is consistent, construct a Turing machine T that does not halt, yet ZFC does not prove that $\ulcorner T \urcorner$ does not terminate.

(4) Consider the problem of deciding, given a two-tape Turing machine T and an input word w, whether during the run of T on input w, T ever writes a symbol on the second tape that's not blank. Show that this problem is not decidable.

(5) Let κ be an infinite cardinal. Recall that H_{κ^+} is the collection of all sets x whose transitive closure has cardinality at most κ. Define a *code* to be a pair $\langle R, \xi\rangle$ such that $R \subseteq \kappa \times \kappa$, $\langle \kappa, R\rangle$ is an extensional, well-founded structure and $\xi < \kappa$. Given a code $c = \langle R, \xi\rangle$, let $f_c : \langle \kappa, R\rangle \longrightarrow \langle u_c, \in\restriction u\rangle$ be the Mostowski-isomorphism. Say that the set coded by c is $f_c(\xi)$.
   (a) Show that if c is a code, then the set coded by c is a member of H_{κ^+}.
   (b) Show that for every member of H_{κ^+} is coded by some code.
   (c) Conclude that the cardinality of H_{κ^+} is 2^κ.

(6) Suppose α is an uncountable limit ordinal. Show that the following are equivalent:
   (a) α is regular,
   (b) for every function $f : \alpha \longrightarrow \alpha$, the set $\{\beta < \alpha \mid f``\beta \subseteq \beta\}$ is club in α.
   (c) for every function $f : \alpha \longrightarrow \alpha$, there is a nonzero $\beta < \alpha$ such that $f``\beta \subseteq \beta$.

(7) Let $\langle \aleph_\alpha \mid \alpha \in \mathrm{On}\rangle$ be the monotone enumeration of the infinite cardinals.
   (a) Assume that κ is a regular limit cardinal. Show that $\kappa = \aleph_\kappa$.
   (b) Prove in ZFC that there is a club class of cardinals λ such that $\lambda = \aleph_\lambda$.

(8) Recall that a nonempty set $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra if $\mathcal{A}$ is closed under complements (i.e., if it is the case that whenever $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$) and under countable unions (i.e., if $A_n \in \mathcal{A}$, for all $n < \omega$, then $\bigcup_{n<\omega} A_n \in \mathcal{A}$).
   Now let $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(X)$. In order to construct the least σ-algebra $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$, define a sequence $\langle \mathcal{A}_i \mid i \leq \theta\rangle$ by letting $\mathcal{A}_0 = \mathcal{A}$, letting $\mathcal{A}_{i+1}$ consist of all sets of the form $\bigcup_{n<\omega} A_n$, where for all $n < \omega$, $A_n \in \mathcal{A}_i$ or $X \setminus A_n \in \mathcal{A}_i$, and letting $\mathcal{A}_\lambda = \bigcup_{i<\lambda} \mathcal{A}_i$ for limit ordinals λ. Find a θ such that $\mathcal{A}_\theta$ provably is a σ-algebra, regardless of the specific $\mathcal{A}$ and X chosen.