Logic Qualifying Exam
Three Parts
May 25, 2021

Part Zero (16 points)

Answer each of the following questions fully. None should take more than a few paragraphs, and some may need less than that. It is not advisable to spend more than 30 minutes on Part Zero.

1. Give a detailed proof that there is no largest cardinal. Cite axioms from **ZFC** as needed.
2. Among the following three sets, which are m-reducible (alternatively, 1-reducible) to which others? (This is really six questions, but your justifications may be quite brief.)

   $$\mathrm{Th}(\mathcal{N}) \qquad \mathrm{Cn}(PA) \qquad \text{the Halting Problem } K.$$

   Here $\mathcal{N} = (\mathbb{N}, +, \cdot)$ is the structure of the natural numbers under addition and multiplication, and $\mathrm{Cn}(PA)$ is the set of theorems provable from the (decidable) set PA, which consists of the axioms of Peano Arithmetic.
3. Let $\mathcal{A}$ be a substructure of $\mathcal{B}$. Suppose that for every finite tuple $a_1, \ldots, a_n$ of elements of $\mathcal{A}$ and for every $b \in \mathcal{B}$, there is an automorphism f of $\mathcal{B}$ with $f(a_i) = a_i$ for each $i \leq n$ and $f(b) \in \mathcal{A}$. Prove that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$.
4. Prove that it is consistent with the theory of the structure $(\mathbb{N}, 0, 1, <, +, \cdot)$ for a number to have infinitely many prime factors.

Part One (42 points)

Do THREE of the following six problems (which continue on the next page). All syntax is first order—with equality, =, a logical symbol (hence always part of any of the languages L considered). Please justify your answers with full proofs, where you may quote well-known results by name (without proof).

1. Let $\varphi(x, y)$ be an L-formula in the free variables, x and y. Given an L-structure M and an element $c \in M$, the parametric formula $\varphi(x, c)$ defines a set in M, denoted by $\varphi(M, c)$.

   (a) Prove: if T is a complete L-theory such that, for every model $M \models T$ and every element $c \in M$, the set $\varphi(M, c)$ is finite, then there is a uniform finite bound on the cardinality of all those sets $\varphi(M, c)$.

   (b) Prove or disprove the converse: when there is such a uniform bound, all the sets $\varphi(M, c)$ are finite (in all $M \models T$).

2. (a) Prove or disprove: the ordering of the rationals embeds elementarily into any non-trivial dense linear order without endpoints.

   (b) Prove or disprove: any two non-trivial dense linear order without endpoints are elementarily equivalent.

3. Let L_P be the language whose only non-logical symbol is a unary predicate P. Let T_P be the L_P-theory axiomatized by the single axiom $\forall x P(x) \vee \forall x \neg P(x)$. Let T_P^∞ be the theory of all infinite models of T_P.

   (a) How to axiomatize T_P^∞?

   (b) How many models, up to isomorphism, does T_P have in any given cardinality (finite or infinite)?

   (c) How many completions does T_P^∞ have?

   (d) How many models does each of T_P^∞'s completions have in any given infinite cardinality?

   (e) Prove that every embedding of non-empty models of T_P^∞ is elementary.

4. Suppose $L_g = (1, \cdot)$ with the only nonlogical symbols, a constant symbol 1 and a binary function symbol $\cdot$.

   Prove or disprove: the theory of groups is axiomatizable by universal L_g-sentences, i.e., L_g-sentences such that, when they are written in prenex normal form, the only quantifiers are $\forall$.

5. Suppose $L_\mathbb{Z} = (0, +, -)$ with a constant, 0, and two binary function symbols, + and −.

   Consider the additive group of the integers, $\mathbb{Z}$, as an $L_\mathbb{Z}$-structure with the standard interpretations.

   (a) Prove: all substructures of $\mathbb{Z}$ are elementarily equivalent to $\mathbb{Z}$, and there are infinitely many.

   (b) Prove: no proper substructure (in fact, sub*model*) of $\mathbb{Z}$ is elementary.

   (c) Prove or disprove: therefore the complete theory of $\mathbb{Z}$ has quantifier elimination.
   (Part One continues on the next page.)

6. Suppose L_E is a language whose single non-logical symbol is a binary relation symbol E. Consider the L_E-theory T of all L_E-structures in which E defines an equivalence relation with two infinite E-classes.

   (a) Write down an L_E-axiomatization of T.

   (b) How many non-isomorphic countable models does T have?

   (c) How many non-isomorphic models does T have of cardinality $\aleph_1$?

   (d) Conclude what you can about the completeness of T.

Part Two (42 points)

Do THREE of the following six problems.

1. Show (in **ZF**): for a limit ordinal λ, the cofinality of λ is the unique regular cardinal κ such that there is an increasing function from κ to λ whose range is unbounded in λ.

2. Show (in **ZF**): if λ is a singular ordinal, then there are a stationary subset S of λ and a function $f : S \longrightarrow \lambda$ such that for every $\alpha \in S$, $f(\alpha) < \alpha$, and such that f is not constant on any stationary subset of S.

   Is this true if λ is regular? Why or why not?

3. Suppose $A \subseteq \omega$ is recursively enumerable, and for every $e \in A$, $W_e = \omega$, that is, φ_e is total. Show that there is an $x \in \omega$ such that $W_x = \omega$ and $\varphi_x \neq \varphi_e$ for every $e \in A$.

4. Suppose **ZFC** is consistent. Show that there is a Turing program φ_e that never terminates, on any input, but such that **ZFC** does not prove that it never terminates.

5. Let T be a deductively closed, consistent and decidable set of sentences (say in a first order language with finitely many symbols). Show that there is a complete, consistent, decidable $T' \supseteq T$.

6. For a model M of Peano Arithmetic, let $D(M)$ be the set of $a \in M$ such that a is definable in M without parameters (that is, for some formula $\varphi(x)$, $M \models \varphi(a)$, but $M \models \neg\varphi(b)$ for all $b \in M$ with $b \neq a$). As is common, we will say that a set $A \subseteq M$ is *definable in M* if there is a formula $\varphi(x)$ such that $A = \{a \mid M \models \varphi(a)\}$. It is said to be *definable in M using parameters* if for some $\vec{b} \in M$ and some formula $\varphi(x, \vec{y})$, $A = \{a \mid M \models \varphi(a, \vec{b})\}$.

   (a) Show that there is a (countable) model M of Peano Arithmetic such that $D(M)$ is not definable in M (even allowing the use of parameters). In fact, if the standard model $\mathcal{N}$ is a proper elementary submodel of M, then M is as wished.

   (b) Show that every model N of Peano Arithmetic has a countable elementary submodel M such that $D(M)$ is definable in M.