NAME :

THE DEPARTMENT OF MATHEMATICS
THE CUNY GRADUATE CENTER
REAL ANALYSIS QUALIFYING EXAM

Do any EIGHT of the following twelve problems, and put a check below next to each of the problems you want to be graded.

(1) ____
(2) ____
(3) ____
(4) ____
(5) ____
(6) ____
(7) ____
(8) ____
(9) ____
(10) ____
(11) ____
(12) ____

- You have three hours.
- Use only one side of each sheet. Attach extra sheets if needed.

Date: August 3, 2016.

(1) Let X, Y, and Z be topological spaces and let $z = f(x, y)$ be a mapping from $X \times Y$ to Z. We say that f is continuous in x if for each $y \in Y$, the mapping $g \colon X \to Z$ given by $g(x) = f(x, y)$ is continuous. We define f being continuous in y analogously. Prove the following statement if it is true or give a counterexample if it is false. If f is continuous in x and continuous in y, then it is continuous.

(2) Is there a metric d on the set of natural numbers $\mathbb{N}$ such that the metric space $(\mathbb{N}, d)$ is bounded, complete, and the topology induced by d agrees with the standard topology of $\mathbb{N}$? What if $(\mathbb{N}, d)$ is assumed to be a subspace of $\mathbb{R}^n$ for some $n \in \mathbb{N}$? Justify your claims.

(3) Let $C[0,1]$ be the Banach space of real-valued continuous functions on $[0,1]$ endowed with the sup-norm. Let $C^1[0,1]$ be the subspace of $C[0,1]$ that consists of all continuously differentiable functions on $[0,1]$ (at the end points of $[0,1]$ one takes one-sided derivatives). Prove that the derivation operator $T(f) = f'$ from $C^1[0,1]$ to $C[0,1]$ is not bounded. However, prove that T has closed graph.
Hint : Use the Fundamental Theorem of Calculus.

(4) Prove that the following family of functions on [0, 2] is normal, i.e., every sequence has a uniformly convergent subsequence.

$$\mathcal{F} = \left\{ \sin\left(\left(\frac{x}{e}\right)^{n^2}\right) : n \in \mathbb{N} \right\}.$$

(5) Let a and b be real numbers satisfying $a < b$. Prove that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of discontinuity points of f has Lebesgue measure 0.

(6) Let $(X, \mathcal{M}, \mu)$ be a measure space (in which the measure μ is positive), and let $\{f_n\}_{n=0}^{\infty}$ be a sequence of measurable real-valued functions. Let

$$C := \{\, x \in X \ : \ \{f_n(x)\}_{n=0}^{\infty} \subset \mathbb{R} \text{ converges} \,\}.$$

Prove that C is measurable.

(7) Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces in which the measures μ and ν are positive, let $\lambda = \mu \times \nu$ be the product measure on $X \times Y$, and let μ^*, ν^* and λ^* be the induced outer measures on the power sets of X, Y and $X \times Y$ respectively. Prove that $\lambda^*(A \times B) = \mu^*(A) \times \nu^*(B)$ for any (not necessarily measurable) subsets $A \subset X$ and $B \subset Y$.

(8) Prove that an arbitrary collection of pairwise disjoint Lebesgue measurable subsets of $\mathbb{R}$, each of which has positive measure, is at most countable.

(9) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f \in L^p \cap L^\infty$. Prove that $f \in L^q$ for any $q > p$ and that $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$.

(10) (a) Prove that

$$\int_{-\infty}^{\infty} x^{2n}\, e^{-x^2}\, \mathrm{d}x = \frac{(2n)!}{n!}\,\frac{\sqrt{\pi}}{4^n}$$

by differentiating the identity

$$\int_{-\infty}^{\infty} e^{-tx^2}\, \mathrm{d}x = \sqrt{\frac{\pi}{t}}\ .$$

Justify your steps!

(b) Prove that for $\alpha > 0$

$$\int_{-\infty}^{\infty} e^{-x^2} \cos \alpha x\, \mathrm{d}x = \sqrt{\pi} e^{-\alpha^2/4}\ .$$

(11) Consider K a closed convex subset of a Hilbert space $\mathcal{H}$, that is

$$\forall x, y \in K \text{ and } 0 \leq \lambda \leq 1,\ \lambda x + (1-\lambda) y \in K\ .$$

Prove that there exists a unique element y whose norm is minimal.
Hint : The parallelogram law is useful.

(12) Consider μ and ν, two σ-finite positive measures on $(X, \mathcal{M})$ with $\nu \ll \mu$. Let $\rho = \mu + \nu$.

(a) Show that $\nu \ll \rho$ and there exists f such that $0 \leq f \leq 1$ μ-a.e. and

$$\nu(A) = \int_A f \, \mathrm{d}\rho \qquad \forall A \in \mathcal{M} .$$

(b) Conclude from part a) that

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \frac{f}{1-f} \quad \rho - a.e.$$