Name (Print clearly): ____________________

Real Variables Qualifying Exam
The Graduate Center, CUNY, August 2017

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
TOTAL	60	

Instruction:

(1) This exam contains nine problems, but at most six problems will be graded. Please clearly list problems you wish to be graded here:

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(2) Use only one side of each sheet. Do at most one problem on each page;

(3) Write your name on each page; if you include additional pages, write your name on as well.

(4) Justify your answers. Where appropriate, state without proof the results you are using;

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Problem 1. Let $f_n : [0, 1] \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of monotone continuous functions which converges pointwise to a continuous function f. Show that the convergence is uniform on $[0, 1]$.

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Problem 2.

Let m be Lebesgue measure on $\mathbb{R}$. Suppose that $A \subset \mathbb{R}$ is a Lebesgue measurable set such that $m(A) > 0$. Prove that for every $\varepsilon \in (0,1)$, there is an interval I such that $m(A \cap I) \geq (1-\varepsilon)m(I)$.

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Problem 3.

Let $f \in L^1(\mathbb{R})$, and g be a bounded measurable function on $\mathbb{R}$, prove that

$$\lim_{t\to 0} \int_{\mathbb{R}} g(x)\{f(x) - f(x+t)\}dx = 0.$$

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Problem 4.

Let H be a Hilbert space with the norm $\|\cdot\|$, and $E : H \to \mathbb{R}$ be a linear operator such that $\|E\| = 1$, then there exists a unique $y \in H$ such that $\|y\| = 1$ and $E(y) = 1$. (Recall that the norm of a bounded linear operator E is defined as $\|E\| = sup_{\|x\|=1}|E(x)|$.)

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Problem 5.

Let $E = (1, \infty) \subset \mathbb{R}$, and $f(x) = \frac{1}{\sqrt{x}(1+\ln(x))}$ for $x > 1$. Determine all values $p \in [1, \infty]$ such that $f \in L^p(E)$.

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Problem 6.

Compute the following limits and justify your answer:

(1)

$$\lim_{n\to\infty} \int_0^\infty (1 + \frac{x}{n})^{-n} \sin(\frac{x}{n}) dx \quad ;$$

(2)

$$\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} dx \quad .$$

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Problem 7.

Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) = 1$. Show that for any given $f \in L^p(X)$, the function $\|f\|_r$ is non-decreasing in r on $(0, p]$.

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Problem 8.

Let$(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$.

(1) Show that the mapping defined for functions $f, g \in \mathcal{A}$ by

$$d(f,g) = \int_X \frac{|f-g|}{1+|f-g|} d\mu$$

is a metric on the space of measurable functions.

(2) Prove that a sequence $\{f_n\}_{n\geq 1}$ converges in measure to f if and only if $d(f_n, f) \to 0$.

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Problem 9.

Consider Lebesgue measure on $[0,1]$, and let $f_n = n\chi_{(0,1/n)}$ where χ_A stands for the characteristic function of the set A. Prove that $f_n \to 0$ a.e. and in measure, but f_n does not converge weakly in $L^p([0,1])$ for any $1 \le p \le \infty$.