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# Real Variables Qualifying Exam
# The Graduate Center, CUNY, August 2020

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| TOTAL | 50 | |

**Instructions:**

(1) The exam will be conducted while on Zoom. Have a camera turned on so that you are visible to the proctor for the entire exam.

(2) The exam will be 2 hours and 30 minutes followed by a short discussion part having to do with the questions that were asked on the exam (held in a breakout room). When you complete the exam, scan the pages you want graded to PDF (camera app is fine) and send it to your proctor. The discussion will occur after the proctor receives your scanned answers. The intent of the discussion is to allow you to address incomplete answers.

(3) This exam contains eight problems, but at most five problems will be graded. Please clearly list these here or on the first page scanned:

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(4) If possible, print out this exam and work on individual problems on the printed sheets. Use of blank white paper is also acceptable. Do at most one problem on each page, and be sure to write your name on each.

(5) Justify your answers. Where appropriate, state without proof the results you are using. Each part of a problem counts equally.

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**Problem 1.** Let $E$ be a subset of a Hilbert space $\mathcal{H}$. Show that $(E^{\perp})^{\perp}$ is the smallest closed subspace containing $E$. Remember that $F^{\perp} = \{x \in \mathcal{H} : (x, y) = 0 \text{ for any } y \in F\}$.

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**Problem 2.**

Consider $m$, the Lebesgue measure on $\mathbb{R}$.

(a) Prove that the Lebesgue measure is translation-invariant on Borel sets. More precisely, for any $x \in \mathbb{R}$ and Borel sets $B \in \mathcal{B}(\mathbb{R})$, we have $m(x + E) = m(E)$, where $E + x = \{y + x : y \in E\}$.

(b) Argue that the translation invariance extends to all Lebesgue measurable sets $E$.

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**Problem 3.**

Let $X = [0,1]$ equipped with the Lebesgue measure.

(a) Let $f_n(x) = \cos(2\pi n x)$. Show that $f_n \to 0$ weakly in $L^2$. Show also that $f_n$ does not converge to 0 a.s. nor in measure.

(b) Let $f_n(x) = n\chi_{[0,1/n]}$. Show that $f_n \to 0$ a.e. and in measure. However, $f_n$ does not converge to 0 weakly in $L^p$ for any $1 \leq p < \infty$.

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**Problem 4.**

Let $X$ be the set of real-valued sequences $\{x_n\}$ which only take finitely many values, i.e.,

$$\{x_n \in \mathbb{R} : \ n \in \mathbb{N}\}$$

is finite. Prove that $X$ is dense in $\ell^\infty$.

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**Problem 5.**

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions converging pointwise. Prove that there is an $M \in \mathbb{R}$ and an open interval $(a, b)$ such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in (a, b)$.

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**Problem 6.**

Prove that if $X$ is a compact metric space, then the space $C(X)$ of continuous functions $X \to \mathbb{R}$ endowed with the uniform norm is separable. You can use without proof that $X$ is separable.

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**Problem 7.**

Let $a \geq 0$. Calculate

$$I(a) = \int_0^\infty e^{-ax} \frac{sin(x)}{x} dx.$$

You must justify your steps.

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**Problem 8.**

Let $f$ be a continuous function on $\mathbb{R}$, and $g$ be a measurable function on $\mathbb{R}$. If for any set $Z \subset \mathbb{R}$ of zero measure, $f^{-1}(Z)$ is measurable, prove that $g\big(f(x)\big)$ is a measurable function.