Name (Print clearly): ______________________

Real Variables Qualifying Exam
The Graduate Center, CUNY, May 2013

Problem	Points	Score
1	15	
2	15	
3	15	
4	20	
5	20	
6	20	
7	20	
8	20	
TOTAL	110	

Instruction:

(1) This exam contains two parts. In part 1, there are three problems, but at most two of them will be graded. In part 2, there are five problems, and at most four of them will be graded. Please clearly list problems you wish to be graded here: ______________________

(2) Use only one side of each sheet. Do at most one problem on each page;

(3) Write your name on each page;

(4) Justify your answers. Where appropriate, state without proof the results you are using;

Name (Print clearly): ______________________________

PART I: Problem 1.

Suppose $x_1 > 0$, and $x_{n+1} = \frac{1}{2+x_n}$ for $n = 1, 2, \cdots$. Prove that the sequence $\{x_n\}$ converges and find its limit. Justify your steps.

Name (Print clearly): ____________________

Problem 2.

Let H be a Hilbert space with the induced norm $\|\cdot\|$, and let E be a nonempty closed convex subset of H. Prove that there exists a unique element $x \in E$ such that

$$\|x\| = \inf_{y \in E} \|y\|.$$

Name (Print clearly): ____________________

Problem 3.

Let (E, μ) be a measure space with $\mu(E) = 1$. Suppose that there exists some $q > 0$ such that $f \in L^q(E)$. Prove that

$$\lim_{p \to 0} \|f\|_p = \exp\{\int_E \log|f| d\mu\}.$$

Here we adopt the convention that $\exp(-\infty) = 0$.

Name (Print clearly): ______________________________

Part II: Problem 4.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable on $[0, 1]$, and it satisfies $f(x + y) = f(x)f(y)$ for $\forall x, y \in [0, 1]$ with $f(0) = 1$. Prove that $f(x) = e^{ax}$, for some constant a

Name (Print clearly): ______________________________

Problem 5.

Let $f \in L^2([0,1])$ and $\int_0^1 f^2 \leq 1$.

(1) Show that for each $t \in (0,1]$, we have $\int_0^t |f| \leq \sqrt{t}$;

(2) Show that $\lim_{t \to 0}\{t^{-\frac{1}{2}} \int_0^t |f|\} = 0$.

Name (Print clearly): ______________________________

Problem 6.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $0 < a < b$, and $\lim_{n\to\infty} f(nx) = 0$ for every $x \in [a, b]$. Show that $\lim_{x\to\infty} f(x) = 0$.
(Hint: For a given $\varepsilon > 0$, consider the sets $E_k = \{x \in [a, b] : |f(nx)| \leq \varepsilon \text{ for all } n > k\}$.)

Name (Print clearly): ______________________________

Problem 7.

Suppose $\{f_n\}$ is a sequence of measurable functions on $[0,1]$ which converges to zero almost everywhere. Prove that there exists a sequence $\{t_n\}$ of real numbers such that $\sum_{n=1}^{\infty} |t_n| = \infty$ and

$$\sum_{n=1}^{\infty} |t_n f_n(x)| < \infty \qquad a.e. \ \ x \in [0,1].$$

Name (Print clearly): ______________________________

Problem 8.

Suppose $f \in L^\infty(\mathbb{R})$, where the measure on $\mathbb{R}$ is the usual Lebesgue measure. Prove that

$$\lim_{n\to\infty} \{ \int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2} dx \}^{\frac{1}{n}}$$

exists and equals $\|f\|_\infty$.