Department of Mathematics
The CUNY Graduate Center
Real Analysis Qualifying Exam
Spring 2014

Your name: ________________________________

<table>
<thead>
<tr>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>B4</th>
<th>B5</th>
<th>B6</th>
<th>B7</th>
</tr>
</thead>
</table>

Instructions

- The exam has two parts. Answer two questions from Part A and five questions from Part B. Only two + five questions will contribute to your score.
- In the above table, check the questions from each part that you would like to be graded.
- Use only one side of each sheet. Attach extra sheets if necessary.
- You have 3 hours to complete your work.

Conventions

- The terms “measurable,” “measure,” “integrable,” and “almost everywhere (a.e.)” in $\mathbb{R}^n$ always refer to the $\sigma$-algebra of Lebesgue measurable sets and Lebesgue measure $m$.
- All measures are positive.
- If $(X, \mu)$ is a measure space, we denote by $L^p(X, \mu)$ the space of measurable functions $f : X \to \mathbb{C}$ for which

$$
\|f\|_p = \begin{cases} 
\left( \int_X |f|^p \, d\mu \right)^{1/p} & 1 \leq p < \infty \\
\text{ess sup} |f| & p = \infty
\end{cases}
$$

is finite. When $X \subset \mathbb{R}^n$ and $\mu = m$, we simplify this notation to $L^p(X)$. 
PART A. THEOREMS/DEFINITIONS  Answer any two of the following three questions.

A1. Define a *separable* metric space. For what values of $1 \leq p \leq \infty$ is the space $L^p(\mathbb{R})$ separable? Briefly explain why.
A2. Carefully state *Fubini's theorem* on integration on the product of two measure spaces. What does this theorem imply about the 2-dimensional Lebesgue measure of the graph of a measurable function $f : \mathbb{R} \to \mathbb{R}$?
A3. Assuming the Axiom of Choice, explain how one can prove that R (in fact every subset of R with positive measure) has Lebesgue non-measurable subsets.
PART B. PROBLEMS Solve any five of the following seven problems.

B1. Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is continuous. Show that

$$\lim_{n \to \infty} \int_0^1 n x^n f(x) \, dx = f(1).$$
**B2.** Suppose $E \subseteq \mathbb{R}$ is measurable with $m(E) > 1$. Prove that there exist $x, y \in E$ such that $x - y$ is an integer.
B3. Prove that there is no sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{n \to \infty} f_n(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}.$$
B4. Let $(X, \mu)$ be a measure space, $f \in L^p(X, \mu)$, $g \in L^q(X, \mu)$, and $h \in L^r(X, \mu)$, where $1 < p, q, r < \infty$ and $p^{-1} + q^{-1} + r^{-1} = 1$. Show that $fgh \in L^1(X, \mu)$ and
\[
\|fgh\|_1 \leq \|f\|_p \|g\|_q \|h\|_r.
\]
B5. Suppose \((X, \mu)\) is a measure space and \(\{f_n\}\) is a sequence in \(L^p(X, \mu)\) for some \(1 \leq p < \infty\).

(i) Prove that if \(\sum_{n=1}^{\infty} \|f_n\|_p\) converges, then \(f_n \to 0\) a.e. in \(X\).

(ii) Show by an example that the assumption in (i) cannot be relaxed to \(\|f_n\|_p \to 0\).
B6. Recall that $f : [a, b] \to \mathbb{R}$ is a Lipschitz function if there is a constant $C > 0$ such that $|f(x) - f(y)| \leq C |x - y|$ for all $x, y \in [a, b]$. Show that the following conditions are equivalent:

(i) $f$ is Lipschitz on $[a, b]$.

(ii) $f$ is absolutely continuous on $[a, b]$ and $f' \in L^\infty[a, b]$. 
B7. Suppose $X, Y$ are Banach spaces and $T : X \to Y$ is linear and surjective. If there is a constant $C > 0$ such that $\|T(x)\| \geq C$ for every unit vector $x \in X$, show that $T$ is bounded.