The Department of Mathematics
The CUNY Graduate Center

Real Analysis Qualifying Exam

18 May 2015

Your name: ______________________________

Do any 8 of the following 12 problems, and put a check below next to each of the problems you want graded.

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______
7. ______
8. ______
9. ______
10. ______
11. ______
12. ______

Further Instructions/Information

- *Use only one side of each sheet.* Attach extra sheets if necessary, use only one side of each of those sheets, and make sure your name is on each of those sheets.
- You have three hours.
- Lebesgue measure is always denoted m.

1. (Do both parts)

   (a) Let $(X, \mathcal{M}, \mu)$ be a complete measure space and suppose $f, g : X \to [0, \infty]$ are functions such that f is measurable and $f = g$ almost everywhere. Prove that g is measurable.

   (b) Let $X \subset \mathbb{R}^n$ be bounded. If every continuous function $f : X \to \mathbb{R}$ is uniformly continuous, prove that X is compact.

2. Show that for every Lebesgue measurable set $E \subset [0,1]$ one has

$$\int_E x\, dm \geq \frac{1}{2}(m(E))^2.$$

3. Let $\mathcal{F}$ be an equicontinuous family of maps from a compact metric space X into $\mathbb{R}$. Prove that if a sequence $f_n \in \mathcal{F}$ converges pointwise to a function $f : X \to \mathbb{R}$ then the convergence is uniform.

4. Use the Baire category theorem to prove there exists a continuous function $f : [0, 1] \to \mathbb{R}$ which is not monotone on any interval of positive length.

5. For each $r \in \mathbb{Q}$ consider the sequence of continuous functions defined by

$$f_{n,r}(x) := \begin{cases} 0 & \text{if } x < r - \frac{1}{n+1} \\ (n+1)(x-r)+1 & \text{if } r - \frac{1}{n+1} \leq x \leq r \\ -(n+1)(x-r)+1 & \text{if } r < x < r + \frac{1}{n+1} \\ 0 & \text{if } x > r + \frac{1}{n+1}. \end{cases}$$

Since $(f_{n,r})$ is doubly indexed by countable sets, it can be viewed as a sequence indexed by $m \in \mathbb{Z}^+ := \{1, 2, 3, \dots\}$. Assuming this viewpoint compute $\limsup_{m\to\infty} f_m$ and $\liminf_{m\to\infty} f_m$.

6. Use Fubini's theorem to evaluate the integral

$$\int_0^\infty \frac{e^{-x} - e^{-2x}}{x}\, dx$$

and justify your steps. (Hint: One can express $\frac{1}{x}$ as an integral of an exponential function involving x and y.)

7. Let T be a linear operator on a complex vector space X with inner product $\langle \cdot, \cdot \rangle$. If $\langle T(x), x \rangle = 0$ for all $x \in X$ prove that $T = 0$.

8. Prove or disprove: the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & \text{if} \quad x = 0; \\ x^2 \sin(1/x^2) & \text{otherwise} \end{cases}$$

is absolutely continuous.

9. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers in $[0,1]$ and define

$$f(x) = \sum_{n=1}^{\infty} 2^{-n}|x - r_n|^{-1/2}.$$

Prove that $f(x) < \infty$ for almost every $x \in [0,1]$ even though f is unbounded on every open interval contained in $[0,1]$.

10. Suppose X is a compact metric space and $f : X \to X$ is an isometry in the sense that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that $f(X) = X$.

11. Give an example of a sequence of Lebesgue integrable functions $f_n : \mathbb{R} \to [0, \infty)$ which converge uniformly to a function $f : \mathbb{R} \to [0, \infty)$ but satisfy

$$\int_{\mathbb{R}} f \, dm < \liminf_{n \to \infty} \int_{\mathbb{R}} f_n \, dm.$$

12. (Do both parts)

(a) Show that $\Gamma(\lambda) := \int_0^\infty e^{-x} x^{\lambda-1}\, dx$ exists for each $\lambda > 0$.

(b) Prove that $\Gamma(\lambda) = \lim_{n\to\infty} \int_0^n (1 - \frac{x}{n})^n x^{\lambda-1}\, dx$.