Your name: ____________________________________________

Do any 8 of the following 12 problems, and put a check below next to each of the problems you want graded.

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______
7. ______
8. ______
9. ______
10. ______
11. ______
12. ______

Further Instructions/Information

- Use only one side of each sheet. Attach extra sheets if necessary, use only one side of each of those sheets, and make sure your name is on each of those sheets.

- You have three hours.

- Lebesgue measure is always denoted $m$. 
1. (Do both parts)

(a) Let \((X, \mathcal{M}, \mu)\) be a complete measure space and suppose \(f, g : X \rightarrow [0, \infty]\) are functions such that \(f\) is measurable and \(f = g\) almost everywhere. Prove that \(g\) is measurable.

(b) Let \(X \subseteq \mathbb{R}^n\) be bounded. If every continuous function \(f : X \rightarrow \mathbb{R}\) is uniformly continuous, prove that \(X\) is compact.
2. Show that for every Lebesgue measurable set $E \subset [0, 1]$ one has

$$\int_E x \, dm \geq \frac{1}{2} (m(E))^2.$$
3. Let $\mathcal{F}$ be an equicontinuous family of maps from a compact metric space $X$ into $\mathbb{R}$. Prove that if a sequence $f_n \in \mathcal{F}$ converges pointwise to a function $f : X \to \mathbb{R}$ then the convergence is uniform.
4. Use the Baire category theorem to prove there exists a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ which is not monotone on any interval of positive length.
5. For each $r \in \mathbb{Q}$ consider the sequence of continuous functions defined by

$$f_{n,r}(x) := \begin{cases} 
0 & \text{if } x < r - \frac{1}{n+1} \\
(n + 1)(x - r) + 1 & \text{if } r - \frac{1}{n+1} \leq x \leq r \\
-(n + 1)(x - r) + 1 & \text{if } r < x < r + \frac{1}{n+1} \\
0 & \text{if } x > r + \frac{1}{n+1}.
\end{cases}$$

Since $(f_{n,r})$ is doubly indexed by countable sets, it can be viewed as a sequence indexed by $m \in \mathbb{Z}^+ := \{1, 2, 3, \ldots\}$. Assuming this viewpoint compute $\limsup_{m \to \infty} f_m$ and $\liminf_{m \to \infty} f_m$. 

6. Use Fubini's theorem to evaluate the integral

$$\int_0^\infty \frac{e^{-x} - e^{-2x}}{x} \, dx$$

and justify your steps. (Hint: One can express $\frac{1}{x}$ as an integral of an exponential function involving $x$ and $y$.)
7. Let $T$ be a linear operator on a complex vector space $X$ with inner product $\langle \cdot, \cdot \rangle$. If $\langle T(x), x \rangle = 0$ for all $x \in X$ prove that $T = 0$. 
8. Prove or disprove: the function \( f : [0, 1] \rightarrow \mathbb{R} \) defined by

\[
f(x) := \begin{cases} 
0 & \text{if } x = 0; \\
x^2 \sin(1/x^2) & \text{otherwise}
\end{cases}
\]

is absolutely continuous.
9. Let \( \{r_n\}_{n=1}^\infty \) be an enumeration of the rational numbers in \([0,1]\) and define

\[
f(x) = \sum_{n=1}^{\infty} 2^{-n} |x - r_n|^{-1/2}.
\]

Prove that \( f(x) < \infty \) for almost every \( x \in [0,1] \) even though \( f \) is unbounded on every open interval contained in \([0,1]\).
10. Suppose $X$ is a compact metric space and $f : X \to X$ is an isometry in the sense that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that $f(X) = X$. 
11. Give an example of a sequence of Lebesgue integrable functions \( f_n : \mathbb{R} \to [0, \infty) \) which converge uniformly to a function \( f : \mathbb{R} \to [0, \infty) \) but satisfy

\[
\int_{\mathbb{R}} f \, dm < \liminf_{n \to \infty} \int_{\mathbb{R}} f_n \, dm.
\]
12. (Do both parts)

(a) Show that $\Gamma(\lambda) := \int_0^\infty e^{-x} x^{-\lambda} \, dx$ exists for each $\lambda > 0$.

(b) Prove that $\Gamma(\lambda) = \lim_{n \to \infty} \int_0^\infty (1 - \frac{x}{n})^n x^{-\lambda} \, dx$. 