Name (Print clearly): ________________________________

Real Variables Qualifying Exam
The Graduate Center, CUNY, May 2017

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Instruction:
(1) This exam contains nine problems, but at most six problems will be graded. Please clearly list problems you wish to be graded here:

(2) Use only one side of each sheet. Do at most one problem on each page;
(3) Write your name on each page; if you include additional pages, write your name on as well.
(4) Justify your answers. Where appropriate, state without proof the results you are using;
Problem 1. Let \((X, \mathcal{A}, \mu)\) be a measure space. Suppose \(\{f_n\}_{n \geq 1}, \{g_n\}_{n \geq 1}, f,\) and \(g\) are all integrable functions. If we have \(f_n \to f\) a.e., and \(g_n \to g\) a.e., \(|f_n(x)| \leq g_n(x)\) for all \(x \in X\) and \(n \geq 1,\) and \(g_n \to g\) a.e. Prove that

\[
\int_X f_n \, d\mu \to \int_X f \, d\mu.
\]
Problem 2.

Let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and $\{e_i\}_{i \geq 1}$ an orthonormal basis. Prove that a sequence $\{x_n\}$ converges weakly to $x$ if and only if $\langle x_n, e_i \rangle \to \langle x, e_i \rangle$ for every basis vector $e_i$ AND $\|x_n\|$ is bounded. Here the norm $\|\cdot\|$ is the one induced by the inner product $\langle \cdot, \cdot \rangle$. 
Problem 3.

Let $1 \leq p < \infty$, and $I = [0, 1]$, find the values of the parameter $t$ such that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{0}^{\varepsilon} f(x) dx = 0,$$

for all $f \in L^p(I)$. 
Problem 4.

Let $f$ be a continuous real-valued function on $[0,1]$.

(i) Suppose that $\int_0^1 f(x)e^{nx}dx = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Prove that $f \equiv 0$ on $[0,1]$.

(ii) Suppose that we have both $f(0) = f(1)$ and $\int_0^1 f(x)e^{2\pi nx}dx = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Prove that $f \equiv 0$ on $[0,1]$. 
Problem 5.

Let $f(x) \in L^1(\mathbb{R}^n)$ where $n \geq 2$. Define $F(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$, where “$x \cdot \xi$” indicates the inner product on $\mathbb{R}^n$. Prove that $F(\xi)$ is uniformly continuous on $\mathbb{R}^n$. 
Problem 6.

Consider the space $L^\infty([0, 1])$ with the Lebesgue measure.

(a) Prove that the evaluation map $f \rightarrow f(0)$ is a bounded linear function on $C([0, 1])$.

(b) Use the Hahn-Banach theorem, to argue there exists a $\ell \in (L^\infty)^*$ such that $\ell(f) = f(0)$ for any $f \in C([0, 1] \subset L^\infty$. 
Problem 7.

Let \((X, \mathcal{A}, \mu)\) a measure space such that \(\mu(X) < \infty\). Consider \(\{f_n\}_{n \geq 1}\) a sequence of functions that converges to \(f\) a.e.. Prove that for any \(\varepsilon > 0\), there exists a set \(E \subset X\) such that \(\mu(E) < \varepsilon\) and \(f_n \to f\) uniformly on \(X \setminus E\).
Problem 8.

For $1 < p < \infty$, let $f \in L^p(\mathbb{R})$. For each positive integer $n$ we define $f_n(x) = f(x - n)$ for all $x$. Prove that $\{f_n\}$ converges weakly to 0 in $L^p(\mathbb{R})$. How about the case $p = 1$? Prove or disprove it.
Problem 9.

All your answers to the following should be justified.

(a) Give an example of a continuous function on $[0, 1]$ which is not of bounded variation on $[0, 1]$ but is absolutely continuous on $[a, 1]$ for each $a \in (0, 1)$;

(b) Give an example of a continuous function of bounded variation on $[0, 1]$ which is not absolutely continuous on $[0, 1]$. 