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Real Variables Qualifying Exam
The Graduate Center, CUNY, May 2017

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
TOTAL	60	

Instruction:

(1) This exam contains nine problems, but at most six problems will be graded. Please clearly list problems you wish to be graded here:

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(2) Use only one side of each sheet. Do at most one problem on each page;

(3) Write your name on each page; if you include additional pages, write your name on as well.

(4) Justify your answers. Where appropriate, state without proof the results you are using;

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Problem 1. Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose $\{f_n\}_{n\geq 1}$, $\{g_n\}_{n\geq 1}$, f, and g are all integrable functions. If we have $f_n \to f$ a.e., and $g_n \to g$ a.e., $|f_n(x)| \leq g_n(x)$ for all $x \in X$ and $n \geq 1$, and $g_n \to g$ a.e.. Prove that

$$\int_X f_n d\mu \to \int_X f d\mu.$$

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Problem 2.

Let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and $\{e_i\}_{i \geq 1}$ an orthonormal basis. Prove that a sequence $\{x_n\}$ converges weakly to x if and only if $\langle x_n, e_i \rangle \to \langle x, e_i \rangle$ for every basis vector e_i **AND** $\|x_n\|$ is bounded. Here the norm $\| \cdot \|$ is the one induced by the inner product $\langle \cdot, \cdot \rangle$.

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Problem 3.

Let $1 \leq p < \infty$, and $I = [0,1]$, find the values of the parameter t such that

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon^t} \int_0^{\varepsilon} f(x)dx = 0,$$

for all $f \in L^p(I)$.

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Problem 4.

Let f be a continuous real-valued function on $[0, 1]$.

(i) Suppose that $\int_0^1 f(x)e^{nx}dx = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Prove that $f \equiv 0$ on $[0, 1]$.

(ii) Suppose that we have both $f(0) = f(1)$ and $\int_0^1 f(x)e^{2\pi nix}dx = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Prove that $f \equiv 0$ on $[0, 1]$.

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Problem 5.

Let $f(x) \in L^1(\mathbb{R}^n)$ where $n \geq 2$. Define $F(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx$, where "$x \cdot \xi$" indicates the inner product on $\mathbb{R}^n$. Prove that $F(\xi)$ is uniformly continuous on $\mathbb{R}^n$.

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Problem 6.

Consider the space $L^\infty([0,1])$ with the Lebesgue measure.

(a) Prove that the evaluation map $f \to f(0)$ is a bounded linear function on $C([0,1])$.

(b) Use the Hahn-Banach theorem, to argue there exists a $\ell \in (L^\infty)^*$ such that $\ell(f) = f(0)$ for any $f \in C([0,1] \subset L^\infty$.

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Problem 7.

Let $(X, \mathcal{A}, \mu)$ a measure space such that $\mu(X) < \infty$. Consider $\{f_n\}_{n\geq 1}$ a sequence of functions that converges to f a.e.. Prove that for any $\varepsilon > 0$, there exists a set $E \subset X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $X \backslash E$.

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Problem 8.

For $1 < p < \infty$, let $f \in L^p(\mathbb{R})$. For each positive integer n we define $f_n(x) = f(x - n)$ for all x. Prove that $\{f_n\}$ converges weakly to 0 in $L^p(\mathbb{R})$. How about the case $p = 1$? Prove or disprove it.

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Problem 9.

All your answers to the following should be justified.

(a) Give an example of a continuous function on $[0, 1]$ which is not of bounded variation on $[0, 1]$ but is absolutely continuous on $[a, 1]$ for each $a \in (0, 1)$;

(b) Give an example of a continuous function of bounded variation on $[0, 1]$ which is not absolutely continuous on $[0, 1]$.