NAME:

THE DEPARTMENT OF MATHEMATICS
THE CUNY GRADUATE CENTER
REAL ANALYSIS QUALIFYING EXAM

Do any SEVEN of the following ten problems, and put a check below next to each of the problems you want to be graded.

(1) ___
(2) ___
(3) ___
(4) ___
(5) ___
(6) ___
(7) ___
(8) ___
(9) ___
(10) ___

- You have three hours.
- Use only one side of each sheet. Attach extra sheets if needed.

Date: May 18, 2018.
(1) For $n \geq 1$, define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = \frac{n \sin(x)}{1 + \sqrt{x^2 + n^2}} + 2e^{\frac{x}{n}}$. Find
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx,
\]
and be sure to justify your answer.
(2) Let $f : [0, 1] \to \mathbb{R}$ be an absolutely continuous function. Prove that $f$ is of bounded variation. Does this remain true if $[0, 1]$ is replaced by $\mathbb{R}$? If so, give a proof, otherwise give a counterexample.
(3) Let $f(x)$ be a real-valued differentiable function defined on $[1, \infty)$ with $f(1) = 1$ and $f'(x) = \frac{1}{x^2 + f^2(x)}$. Prove that

$$\lim_{x \to \infty} f(x)$$

exists and is less than 2.
(4) Let $H$ be an infinite dimensional Hilbert space. Prove that (a) any orthonormal sequence $\{u_n\}$ converges weakly to 0, and (b) the unit sphere $S = \{u \in H : \|u\| = 1\}$ is dense in the unit ball $B = \{u \in H : \|u\| \leq 1\}$ in the weak topology.
Let $X$ be a measure space with (positive) measure $\mu$, and $1 < p < \infty$, $q$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $f \in L^p(X)$ and $g \in L^q(X)$. Prove Hölder’s inequality, namely,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q,$$

and show that the equality holds if and only if there are constants $a$ and $b$, not both zero, such that

$$af^p = bg^q, \text{ a.e.}$$
(6) Let \( X \) be a measure space with (positive) measure \( \mu \), and \( 1 < p < \infty \), \( q \) satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \), and \((L^p)^*\) be the dual space of \( L^p(X) \). Consider the map \( \phi : L^q(X) \to (L^p)^* \) defined as

\[
\phi(g) = \int_X fg \, d\mu,
\]

for fixed \( f \in L^p(X) \). Prove \( \phi \) is an isometry. Does this remain true if \( p = 1 \)? If so, give a proof, otherwise give a counterexample.
(7) (a) Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of non-negative integrable functions on \([0, 1]\). If

\[
\int_0^1 f_n^2(x) \, dx \leq \frac{1}{n^3},
\]

prove that \( f_n \to 0 \) a.e.;

(b) Show that there exists a sequence \( \{g_n\}_{n=1}^{\infty} \) of non-negative integrable functions on \([0, 1]\) with

\[
\lim_{n \to \infty} \int_0^1 g_n^2(x) \, dx = 0,
\]

but the sequence does not converge to 0 a.e..
(8) Let $X$ be a metric space and $C(X)$ be the space of continuous functions on $X$, and \( \{f_n\} \) a sequence in $C(X)$ that converges uniformly on $X$ to some $f \in C(X)$. Show that \( \{f_n\} \) is equicontinuous.
(9) Let \( \{A_n\} \) be a sequence of disjoint measurable subsets of \([0, 1]\) and \([0, 1] = \bigcup_{n=1}^{\infty} A_n\), and let \( B_n \) be a sequence of measurable subsets of \([0, 1]\) such that 

\[
\lim_{n \to \infty} m(B_n \cap A_k) = 0
\]

for each \( k \). Prove that 

\[
\lim_{n \to \infty} m(B_n) = 0.
\]
(10) Let \( f \) be an infinitely differentiable function on \([0, 1]\) and suppose that for each \( x \in [0, 1] \) there is an integer \( n \) (which may depend on \( x \)) such that \( f^{(n)}(x) = 0 \). Then is it true that \( f \) must coincide with some polynomial on \([0, 1]\)? You must justify your claim.