**NAME** :

**THE DEPARTMENT OF MATHEMATICS**
**THE CUNY GRADUATE CENTER**
**REAL ANALYSIS QUALIFYING EXAM**

**Do any SEVEN of the following ten problems, and put a check below next to each of the problems you want to be graded.**

**(1)** _____
**(2)** _____
**(3)** _____
**(4)** _____
**(5)** _____
**(6)** _____
**(7)** _____
**(8)** _____
**(9)** _____
**(10)** _____

- **You have three hours.**
- **Use only one side of each sheet. Attach extra sheets if needed.**

*Date*: May 18, 2018.

(1) For $n \geq 1$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \frac{n\sin(x)}{1+\sqrt{x}n^2} + 2e^{\frac{x}{n}}$. Find

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx,$$

and be sure to justify your answer.

(2) Let $f : [0, 1] \to \mathbb{R}$ be an absolutely continuous function. Prove that $f$ is of bounded variation. Does this remain true if $[0, 1]$ is replaced by $\mathbb{R}$? If so, give a proof, otherwise give a counterexample.

(3) Let $f(x)$ be a real-valued differentiable function defined on $[1, \infty)$ with $f(1) = 1$ and $f'(x) = \frac{1}{x^2+f^2(x)}$. Prove that

$$\lim_{x\to\infty} f(x)$$

exists and is less than 2.

(4) Let $H$ be an infinite dimensional Hilbert space. Prove that (a) any orthonormal sequence $\{u_n\}$ converges weakly to 0, and (b) the unit sphere $S = \{u \in H : \|u\| = 1\}$ is dense in the unit ball $B = \{u \in H : \|u\| \leq 1\}$ in the weak topology.

(5) Let $X$ be a measure space with (positive) measure $\mu$, and $1 < p < \infty$, $q$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $f \in L^p(X)$ and $g \in L^q(X)$. Prove Hölder's inequality, namely,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q,$$

and show that the equality holds if and only if there are constants $a$ and $b$, not both zero, such that

$$af^p = bg^q, a.e.$$

(6) Let $X$ be a measure space with (positive) measure $\mu$, and $1 < p < \infty$, $q$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and $(L^p)^*$ be the dual space of $L^p(X)$. Consider the map $\phi : L^q(X) \to (L^p)^*$ defined as

$$\phi(g) = \int_X f g d\mu,$$

for fixed $f \in L^p(X)$. Prove $\phi$ is an isometry. Does this remain trues if $p = 1$? If so, give a proof, otherwise give a counterexample.

(7) (a) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative integrable functions on $[0,1]$. If

$$\int_0^1 f_n^2(x)dx \leq \frac{1}{n^3},$$

prove that $f_n \to 0 \;\; a.e.$;

(b) Show that there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of non-negative integrable functions on $[0,1]$ with

$$\lim_{n\to\infty} \int_0^1 g_n^2(x)dx = 0,$$

but the sequence does not converge to 0 a.e..

(8) Let $X$ be a metric space and $C(X)$ be the space of continuous functions on $X$, and $\{f_n\}$ a sequence in $C(X)$ that converges uniformly on $X$ to some $f \in C(X)$. Show that $\{f_n\}$ is equicontinuous.

(9) Let $\{A_n\}$ be a sequence of disjoint measurable subsets of $[0,1]$ and $[0,1] = \bigcup_{n=1}^{\infty} A_n$, and let $B_n$ be a sequence of measurable subsets of $[0,1]$ such that

$$\lim_{n\to\infty} m(B_n \cap A_k) = 0$$

for each $k$. Prove that

$$\lim_{n\to\infty} m(B_n) = 0.$$

(10) Let $f$ be an infinitely differentiable function on $[0,1]$ and suppose that for each $x \in [0,1]$ there is an integer $n$ (which may depend on $x$) such that $f^{(n)}(x) = 0$. Then is it true that $f$ must coincide with some polynomial on $[0,1]$ ? You must justify your claim.