Qualifying Exam, Real Analysis
May 2021

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
TOTAL	50	

Instructions:

1. The exam will be conducted while on Zoom. Have a camera turned on so that you are visible to the proctor for the entire exam.
2. The exam will be 2 hours and 30 minutes followed by a short discussion part having to do with the questions that were asked on the exam (held in a breakout room). When you complete the exam, scan the pages you want graded to PDF (camera app is fine) and send it to your proctor. The discussion will occur after the proctor receives your scanned answers. The intent of the discussion is to allow you to address incomplete answers.
3. This exam contains eight problems, but at most five problems will be graded. Please clearly list these here or on the first page scanned:

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4. If possible, print out this exam and work on individual problems on the printed sheets. Use of blank white paper is also acceptable. Do at most one problem on each page, and be sure to write your name on each.
5. Justify your answers. Where appropriate, state without proof the results you are using. Each part of a problem counts equally.

Problem 1
Let $(X, \mathcal{F})$ be a measurable space and μ be a σ-finite measure on $(X, \mathcal{F})$. Prove or disprove the following statements.

(a) If $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, $A_1 \subset A_2 \subset \cdots \subset A_n \subset \ldots$ and $A = \cup_{n=1}^{\infty} A_n$ then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

(b) If $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, $A_1 \supset A_2 \supset \cdots \supset A_n \supset \ldots$ and $A = \cap_{n=1}^{\infty} A_n$ then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

Problem 2
Let $(X, \mathcal{F}, \mu)$ be a measure space and $\mu(X) = \infty$. Show that

$$f_n \to f \ \mu\text{-a.e. and } \ g_n \to g \ \mu\text{-a.e.} \implies f_n g_n \to fg \ \ \mu\text{-a.e.}$$

but

$$f_n \to f \ \mu\text{-a.e. and } \ g_n \to g \ \mu\text{-a.e.} \not\Longrightarrow f_n g_n \to fg \text{ in measure.}$$

Problem 3

(a) Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where $\mu(\{k\}) = 2^{-k}$, $k \in \mathbb{N}$. We consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ given by $f(n,n) = 4^n - 1$, $f(n, n+1) = -2(4^n - 1)$ and $f(m,n) = 0$ otherwise. Show that the conclusion of Fubini's theorem does not hold in this case. Why not ?

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function, and consider its graph $G = \{(x, f(x)) : x \in \mathbb{R}\}$ in $\mathbb{R}^2$. Show that this subset is $\mathcal{B}(\mathbb{R}^2)$-measurable. Prove that $m \times m(G) = 0$.

Problem 4
Consider the set of bounded functions $\mathcal{B}(\mathbb{R}, \mathbb{C})$ valued in $\mathbb{C}$ with the sup-norm, and let $P \subset \mathcal{B}(\mathbb{R}, \mathbb{C})$ be the subset of 2π-periodic functions. Given $N \in \mathbb{N}$, for each $f \in P$ consider the sum:

$$S_N(f) := \sum_{k=-N}^{N} c_k$$

where c_k are the k-th Fourier coefficient of f given by $c_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt$

(a) Show that P is complete with the sup-norm

(b) Show that $S_N : P \to \mathbb{C}$ is a continuous linear form, and its norm is given by

$$\|S_N\| := \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big| \sum_{k=-N}^{N} e^{-ikt} \Big| dt$$

Problem 5

Consider the space ℓ^2 of sequences $x : \mathbb{N} \to \mathbb{R}$ satisfying $\sum_{i=1}^{\infty} x_i^2 < \infty$, with the usual metric. Given a sequence $(a_n)_{n=1}^{\infty}$ of real numbers define the set

$$A := \left\{ (x_1, x_2, \dots) \in \ell^2 \; : \; \sum_{n=1}^{\infty} a_n x_n^2 \leq 1 \right\}$$

(a) Under the assumption that the sequence $(a_n)_{n=1}^{\infty}$ is bounded, show that the set A is not compact.

(b) Under the assumption that $a_n > 0$ for all $n \in \mathbb{N}$ and $a_n \to \infty$ as $n \to \infty$. Show that the set A is compact in ℓ^2.

Problem 6

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \times [a, b] \to \mathbb{R}$ for $-\infty < a < b < \infty$. Suppose that $f(\cdot, t) : X \to \mathbb{R}$ is in $L^1(\mu)$ for each $t \in [a, b]$. Consider the function

$$F(t) = \int_X f(x, t) \mathrm{d}\mu.$$

(a) Suppose there exists $g \in L^1(\mu)$ such that $|f(x, t)| \leq g(x)$ for all x, t. Prove that if $\lim_{t \to t_0} f(x, t) = f(t_0)$ for every x, then $\lim_{t \to t_0} F(t) = F(t_0)$.

(b) Suppose that, for all x, t, the derivative $\frac{\partial f}{\partial t}$ exists and $\left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x)$, for some $g \in L^1(\mu)$. Conclude that F is differentiable on (a, b) and

$$F'(t) = \int_X \frac{\partial f}{\partial t} \mathrm{d}\mu, \qquad \mathrm{t} \in (\mathrm{a}, \mathrm{b})$$

(c) Use this and the fact that $\int_{\mathbb{R}} e^{-tx^2} dx = \sqrt{\frac{\pi}{t}}$ for $t > 0$, to show that

$$\int_{\mathbb{R}} x^{2n} e^{-x^2} dx = \sqrt{\pi} \frac{(2n)!}{4^n n!}.$$

You might want to start with $n = 1$ *to get the pattern.*

Problem 7
Given $\alpha > 1$ and V, W two normed vectors spaces, determine all the functions $f : V \to W$ satisfying:

$$\|f(x) - f(y)\| \leq \|x - y\|^{\alpha} \quad \forall x, y \in V.$$

More generally: Given $\alpha > 1$, V, W two normed vectors spaces and D a dense subset of V, determine all the functions $f : V \to W$ satisfying:

$$\|f(x) - f(y)\| \leq \|x - y\|^{\alpha} \quad \forall x \in V, \;\; \forall y \in D.$$

Problem 8
Consider the weak topology on $C([0,1])$, the Banach space consisting of continuous functions on $[0,1]$ equipped with the sup-norm $\|f\| := \sup_{x\in[0,1]} |f(x)|$.

Let $(f_n)_{n\geq 1}$ be a sequence in $C([0,1])$.

(a) Prove that if $f_n \to f$ weakly then $f_n(x) \to f(x)$ for every $x \in [0,1]$ and $(\|f_n\|)_{n\geq 1}$ is bounded.

(b) Conversely, if $f_n(x) \to f(x)$ for every $x \in [0,1]$ and $(\|f_n\|)_{n\geq 1}$ is bounded, prove that $f_n \to f$ weakly.

Hint: For this second part, you can use that the dual $(C([0,1]))^*$ is the set of finite regular signed Borel measures on $([0,1], \mathcal{B}([0,1]))$.

End of the Exam