(1)

Real Analysis
Hauser - Sullivan

REAL VARIABLES QUALIFYING EXAMINATION

INSTRUCTIONS: Work any 8 problems. Time: three hours.

May 23, 2001

1+

1. ~~Prove that a countable~~ union of sets of measure zero is a set of measure zero.

2. Show that if X is a compact metric space, and $\mathcal{O}$ a collection of open sets which cover X, then there exists an $\epsilon > 0$ such that any ball of radius less than ϵ is contained in at least one of the sets comprising $\mathcal{O}$.

2+

3. ~~State a~~ version of Fubini's theorem, and use it to show that the convolution of two functions in $L^1(-\infty, \infty)$ is in $L^1(-\infty, \infty)$. Conclude by showing that the Fourier transform of the convolution of two L^1 functions is the product of their respective Fourier transforms.

4. State and prove the Lebesgue dominated convergence theorem, assuming the monotone convergence theorem.

5. Show that a continuous linear functional T on a Hilbert space X is of the form $T(x) = (x, y)$, for some $y \in X$.

6. State and prove Weyl's criterion for equidistribution (mod 1) of a sequence of numbers.

7. Show that the functions $\{e^{2\pi i n x}\}$ $(n = 0, \pm 1, \pm 2, \ldots)$ are dense in $L^2[0, 1]$.

3+

8. ~~Discuss in detail the validity of~~ the assertion that in metric spaces, closed and bounded = compact.

9. Show that there exists a transcendental number.

- over -

10. State and prove the Fourier Inversion Theorem for Schwarz functions in $\mathbb{R}^1$.

4

11. Discuss in outline, with pertinent definitions but without proofs, the phenomena of recurrence and transience for random walks on $\mathbb{Z}^2$ and $\mathbb{Z}^3$.

5

12. Suppose $S_1, S_2, \ldots$ are measurable subsets of $\mathbb{R}^1$ and that the sum of their measures is finite. Let A be the set of points in infinitely many of the S_n's. Show that A is of measure zero.

6

13. Suppose X is a complete metric space, and that $A_1, A_2, \ldots$ is a sequence of open dense subsets of X. Show that $\bigcap A_n$ is a dense subset of X. (the Baire Category Theorem).

7

14. In the above problem, suppose $X = [0,1]$. Discuss in detail the cardinality of $\bigcap A_n$.

8

15. Outline a proof of the Central Limit Theorem.

16. Outline a proof that a continuous linear functional $T(f)$ on $C[a,b]$ is given by $T(f) = \int_a^b f(x)\,dG(x)$, where G is a function of bounded variation on $[a,b]$.