Functions of a Real Variable - Qualifying Exam

September, 2005

Do any eight (8) of the following thirteen (13) problems. If you do more than eight clearly mark the eight you wish to be graded.

You do not need to hand in eight correct solutions to pass the examination.

1. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of Lebesgue integrable real-valued function on $[0,1]$ satisfying a) $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$ and b) $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$. Does it follow that $\lim_{n\to\infty} \int_0^1 |f_n(x)|\,dx = 0$? Give either a proof or a counter-example.

2. Let $(X, \mathcal{M}, \mu)$ be a measure triple (or a measure space, depending on your terminology). Suppose $f : X \to \mathbb{R}$ is measurable. If $B \subset \mathbb{R}$ is any Borel subset define $\nu(B) := \mu(f^{-1}(B))$. Show that:

   (a) ν is a measure on the Borel subsets of $\mathbb{R}$; and

   (b) if $g : \mathbb{R} \to [0,\infty)$ is any Borel (measurable) function on $\mathbb{R}$ then

   $$\int_{\mathbb{R}} g\,d\nu = \int_X (g \circ f)\,d\mu.$$

3. A subset A of $\mathbb{R}$ is called discrete if every point of A is an isolated point, that is, for all $x \in A$ there is an $\epsilon = \epsilon_x > 0$ such that $(x-\epsilon, x+\epsilon) \cap A = \{x\}$. Prove that a discrete subset A of $\mathbb{R}$ is necessarily countable.

4. Let $(X, \mathcal{M})$ be a measure space and $\{f_n\}_{n=1}^{\infty}$ a sequence of measurable functions defined on X with values in the extended reals $[-\infty, \infty]$. Prove that $\sup_n f_n$ and $\limsup_{n\to\infty} f_n$ are measurable functions.

5. Let V denote the space of continuous functions $f : [0,1] \to \mathbb{R}$ with norm $f \mapsto \int_0^1 |f|\,dx$ and consider $\mathbb{R}$ as a normed linear space with the usual absolute value as norm. Define $T : V \to \mathbb{R}$ by $T : f \mapsto f(0)$. Is T a bounded operator and, if so, what is $\|T\|$?

6. True or False? For any decreasing sequence $E_1 \supset E_2 \supset E_3 \supset \cdots$ of subsets of $\mathbb{R}^n$ one has $\lim_{n\to\infty} m^*(E_n) = m^*(\bigcap_{n\geq 1} E_n)$. If true give a proof; if false provide a counterexample, give hypotheses which guarantee it is true, and prove you are correct.

7. A metric space (X, d) is *separable* if there is a countable dense subset. Assuming a and b are real numbers with $a < b$ prove that the collection of continuous functions $f : [a,b] \to \mathbb{R}$ with the sup-norm metric is separable.

8. Define polynomial functions $p_n : \mathbb{R} \to \mathbb{R}$ recursively by $p_0(x) = 1$, $p_1(x) = x$ and $p_{n+1}(x) = 2xp_n(x) - p_{n-1}(x)$ for $n > 1$.

   (a) Prove that $p_n(\cos\theta) = \cos(n\theta)$ for all $n \geq 0$.

   (b) Prove that $|p_n(x)| \leq 1$ for $|x| \leq 1$.

   (c) Prove that for $m \neq n$ one has

$$\int_{-1}^{1} \frac{p_n(x)p_m(x)}{\sqrt{1-x^2}}\, dx = 0.$$

9. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be uniformly continuous. Prove that for any two sequences $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ in X satisfying $d(y_n, z_n) \to 0$ one has $|f(y_n) - f(z_n)| \to 0$.

10. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable subsets of $[0,1]$ such that $\lim_{n\to\infty} m(A_n) = 1$. Let $\epsilon \in (0,1)$ be given. Show that there is a subsequence $\{A_{n_j}\}_{j=1}^{\infty}$ of $\{A_n\}$ such that $m(\cap_{j=1}^{\infty} A_{n_j}) > \epsilon$.

11. Prove that any two norms on a finite-dimensional real or complex vector space are equivalent.

12. Let $a, b \in \mathbb{R}$ with $a < b$ and consider $L^2(m_{[a,b]})$, i.e., the vector space of measurable functions $f : [a,b] \to \mathbb{R}$ such that $\int_{[a,b]} f^2\, dm_{[a,b]} < \infty$. Prove that the continuous functions are dense in $L^2(m_{[a,b]})$ (assuming the L^2-metric). (Here $m_{[a,b]}$ denotes the restriction of Lebesgue measure to the closed interval $[a,b]$.)

13. Suppose f is a continuous real-valued function on $[0,\infty)$ and a is a real number. Show that the following two statements are equivalent: i) $\lim_{x\to\infty} f(x) = a$ and ii) for every sequence $\{x_n\}_{n=1}^{\infty}$ of positive numbers such that $\lim_{n\to\infty} x_n = \infty$ we have $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_k) = a$.