REAL VARIABLES QUALIFYING EXAMINATION

INSTRUCTIONS: Work any eight problems. Time: three hours.

September 2, 2009

1. Show that a function which is continuous on a compact metric space X is uniformly continuous on X. Give an example to show that compactness is essential.

2. Suppose $S_1, S_2, \ldots$ are measurable subsets of $\mathbb{R}^1$ and that the sum of their measures is finite. Let A be the set of points in infinitely many of the S_n's. Show that A is of measure zero.

3. Suppose $\psi_1, \psi_2, \ldots$ are linearly independent elements of a Hilbert space H. Prove that the ψ_n's can be orthonormalized. I.e., there exists an orthonormal sequence $\varphi_1, \varphi_2, \ldots$ of elements of H, such that For each N, $\text{span}(\varphi_1, \ldots, \varphi_N) = \text{span}(\psi_1, \ldots, \psi_N)$. Explicitly compute φ_1, φ_2, and φ_3 when $H = \mathrm{L}^2[-1, 1]$ and $\psi_1, \psi_2, \psi_3, \psi_4 \ldots = 1, x, x^2, x^3, \ldots$.

4. Let $\|f\| = \sup_{x \in [0,1]} |f(x)|$ be the sup norm on $C[0, 1]$, the space of real-valued continuous functions on $C[0, 1]$. With this norm, $C[0, 1]$ is a Banach space. Prove or disprove: This norm arises from an inner product on $C[0, 1]$, i.e., $C[0, 1]$ is also a Hilbert space with this norm.

5. State and prove Ascoli's theorem for families of real-valued functions on a compact metric space X. For $X = [0, 1]$, give an example of an equicontinuous family, and of a family which is not equicontinuous.

6. Assuming Beppo Levi's theorem (the Monotone Convergence Theorem), prove Fatou's theorem: if $\{f_n\}$ is a sequence of non-negative functions in L^1 for which $\int f_n \leq M$ and $f_n \to f$ a.e., then $f \in L^1$ and $\int f \leq \liminf \int f_n$.

7. Prove the following strengthened version of the Baire Category Theorem for the space $[0, 1]$: A countable intersection of dense open subsets of $[0, 1]$ has cardinality $\aleph$.

8. let $\|x\|$ denote the distance of x from the nearest integer. Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series, and $0 < \alpha < 1$. Show that the series defining $f(x) = \sum_{n=1}^{\infty} a_n \|nx\|^{-\alpha}$ converges for almost all $x \in R^1$. Hint: By periodicity, it is enough to consider $x \in [0,1]$. Try to use the Monotone Convergence Theorem.

9. Prove the Riemann-Lebesgue Lemma: If $f(x) \in L^1(-\infty,\infty)$, and $\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}\,dx$, then $\hat{f}(y) \to 0$ as $y \to \infty$. Can you give an example for which $\hat{f}(y)$ is not in $L^1(-\infty,\infty)$?

10. Define what it means for a function to be of bounded variation on $[a,b]$, and what it mean for a function to be absolutely continuous on $[a,b]$. Discuss, with examples if possible, the relationship between these two classes. Specifically, must a function of bounded variation be absolutely continuous? Must an absolutely continuous function be of bounded variation?