

REAL VARIABLES QUALIFYING EXAMINATION

INSTRUCTIONS: Work any eight problems. Time: three hours.

May 21, 2007

1. Suppose $p \geq 1$. Outline a proof that $L^p[0, 1]$ is complete.
2. Prove that a linear functional T on a Banach space is continuous if and only if it is bounded, i.e., if and only if $\sup_{\|x\|=1} |T(x)|$ is bounded.
3. In the context of a compact interval $[a, b]$, define bounded variation and absolute continuity. Discuss, without proofs, the differentiability properties of each, in particular additional properties of the derivative in the case of absolute continuity. Give reasons why there are continuous functions which are not of bounded variation.
4. State and prove Hölder's inequality.
5. Suppose E is a subset of \mathbb{R}^1 having positive measure. Define what is meant by a density point of E . State Lebesgue's theorem concerning the measure of the set of such points, and prove that the existence of a density point implies that the difference set of E contains an open interval around the origin (the difference set of E consists of the points of the form $x - y$ with x and y in E).
6. Show that any open cover of a compact metric space X has a *Lebesgue number*, i.e., a positive number ϵ such that any open ball in X of radius ϵ is contained in at least one element of the cover.
7. State the Weyl criterion for equidistribution, and use it to show that if α is irrational, the sequence $\{\alpha, 2\alpha, 3\alpha, \dots\}$ is equidistributed (mod 1).
8. State and prove the Hahn-Banach theorem.

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9. Prove the following Vitali-type lemma: any finite collection C of open intervals has a disjoint subcollection D , such that the union of the intervals in C is covered by the union of the 3-fold dilations of the intervals in D .
10. Prove that there is no function $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ which is continuous precisely at the rationals.
11. Assuming the monotone convergence (Beppo Levi) theorem, show how to prove Fatou's theorem: if $\{f_n\}$ is a sequence of non-negative integrable functions which converge a.e. to a function f , then f is integrable, and

$$\int f \leq \liminf \int f_n.$$

12. State and prove both the convergent case and the divergent case of the Borel-Cantelli lemma.
13. Show that the convolution $(f * g)(y) = \int_{-\infty}^{\infty} f(x)g(y - x) dx$ of two L^1 functions is finite almost everywhere, and is itself L^1 .
14. Prove that the algebra generated by the set $\{1, x^2\}$ is dense in $C[0, 1]$, but not in $C[-1, 1]$.
15. Suppose f is continuous on $[0, 1]$ with $f(1) = 0$. Show that $x^n f(x)$ converges uniformly to 0 on $[0, 1]$ as $n \rightarrow \infty$.
16. Show that not all subsets of the unit circle \mathbb{S}^1 are Lebesgue measurable. Discuss, without proofs, what happens for \mathbb{S}^1 and other spaces, if we retain the requirement of group invariance for a measure, but replace the requirement of countable additivity by the weaker requirement of finite additivity. In particular, discuss without proofs features of the rotation group $SO(3, \mathbb{R})$ acting on \mathbb{S}^2 that are pertinent to this question.
17. State and prove Ascoli's theorem for families of real-valued functions on a compact metric space.