Real Analysis Qualifying Exam May 19, 2008

Do any eight problems. Only the first eight problems in your exam booklet will be graded. Clearly cross out any attempts at solving a problem you do not wish to include amongst the eight to be graded.

1. Prove that
$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$
Hint: Define $f(t) = 4 - t^2$ for $-2 \leq t \leq 2$ and extend the domain to $\mathbb{R}$ by declaring f to be periodic of period 4. Compute the Fourier series of f.

may 21 07 qual

2. Prove that the algebra generated by the functions x^2 and 1 is dense in $C([0,1], \mathbb{R})$ but is not dense in $C([-1,1], \mathbb{R})$. (When X is a topological space, $C(X, \mathbb{R})$ denotes the collection of continuous functions $f : X \to \mathbb{R}$ endowed with the supremum norm.)

3. Let (X, S, μ) be a finite measure space and $\{f_n\}_{n=1}^{\infty}$ integrable functions. Show that if $f_n \to f$ in measure and $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable, then $\int |f_n - f| \, d\mu \to 0$. Show that this needn't be true if $\mu(X) = \infty$.

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space. What is meant by a real valued random variable on $(\Omega, \mathcal{F}, P)$? Suppose that X is a real valued random variable on $(\Omega, \mathcal{F}, P)$. What is meant by the distribution function of X? Give a strong sufficient condition for X to have a probability density function with respect to Lebesgue measure.

5. State the Gram–Schmidt Orthogonalization Theorem. Find four real polynomials that are orthogonal with respect to the measure
$$\mu([-\infty, x)) = \frac{1}{\sqrt{2\pi}} \int_{a}^{x} e^{-x^2/2} \, dx.$$

6. Let (X, S, μ) be a finite measure space . Show that for $1 \leq r \leq s \leq \infty$, $L^s(X, \mathrm{S}, \mu) \subset L^r(X, \mathrm{S}, \mu)$, and that the identity function from L^s into L^r is continuous. Show that this needn't be true if $\mu(X) = \infty$.

7. Let $H = (H, \langle\ ,\ \rangle)$ be a real Hilbert space and let $T : H \to H$ be a self-adjoint linear operator. Prove that any eigenvalue λ of T must be real, i.e., that if $\lambda \in \mathbb{C}$ is an eigenvalue of T then $\lambda \in \mathbb{R}$.

8. Let $X = (X, d)$ be a metric space let $C(X, \mathbb{R})$ denote the collection of continuous functions $f : X \to \mathbb{R}$. Fix any point $p \in X$, and for each point $a \in X$ define the function $f_a : X \to \mathbb{R}$ by

$$f_a : x \mapsto d(x, a) - d(x, p).$$

(a) Prove that $|f_a(x)| \leq d(a, p)$ for any $x \in X$.

(b) Prove that $f_a \in C(X, \mathbb{R})$. $B(X,\mathbb{R})$

(c) Prove that $\|f_a - f_b\| = d(a, b)$ for any $a, b \in X$, where $\|\ \|$ denotes the sup norm.

(d) Define $e : X \to C(X, \mathbb{R})$ by $a \mapsto f_a$, and prove that the closure of $e(X)$ in $C(X, \mathbb{R})$ is complete.

(The result shows that any metric space is isometric (by (c)) to a dense subset of a complete metric space.)

9. Let $(X, \|\cdot\|)$ be a normed space and let $I'' : X \to X''$ be defined by $I''(x)f$:=f(x) for each $x \in X$ and $f \in X'$. Let $\|\cdot\|''$ be the norm on X'' as the dual of X'. Show that $\|I''x\|'' = \|x\|$ for all $x \in X$. Give an example of a normed space $(X, \|\cdot\|)$ for which the range of I'' on X is not all of X''.

10. Let $C[0, 1]$ denote the space of real valued continuous functions on $[0, 1]$. Show that $(C[0, 1], \|\cdot\|)$, where for $f \in C[0, 1]$, $\|f\| := \sup_{x \in [0,1]} |f(x)|$, is a Banach space. Is $(C[0, 1], \|\cdot\|)$ also a Hilbert space under this norm?

11. State and prove the uniform boundedness principle for bounded linear operators on a Banach space.

12. Let $E \subset \mathbb{R}$ be (Lebesgue) measurable and let $r \in \mathbb{R}$. The *Lebesgue density* $\mathrm{den}_E(r)$ of E at the point r is defined to be

$$\mathrm{den}_E(r) := \lim_{h \downarrow 0} \frac{m(E \cap [r - h, r + h])}{2h}$$

provided this limit exists. (Here m denotes Lebesgue measure.) If the limit does not exist at a point r one says that $\mathrm{den}_E(r)$ does not exist. Construct a measurable set $E \subset \mathbb{R}$ such that $\mathrm{den}_E(r)$ does not exist for at least one $r \in \mathbb{R}$.

13. Prove *Lusin's Theorem*: A function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if and only if for each $\epsilon > 0$ there exists a measurable set $E \subset \mathbb{R}$ with $m(E)^{c} < \epsilon$ such that the restriction $f|_{\mathbb{R}\setminus E}$ is continuous. (You may use Egoroff's Theorem.)
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14. State the Monotone Convergence Theorem and Fatou's Lemma. Use the Monotone Convergence Theorem to prove Fatou's Lemma.

15. Show there is no function $f : \mathbb{R} \to \mathbb{R}$ having the irrational numbers as the set of its discontinuities.

16. Let $\mathcal{F}$ be an equicontinuous family of functions between metric spaces X and Y and let $\mathcal{F}^+$ be the family of all pointwise limits of functions in $\mathcal{F}$, i.e., of those f such that there is a sequence $\{f_n\} \subset \mathcal{F}$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$. Prove that $\mathcal{F}^+$ is also an equicontinuous family of functions.