TOPOLOGY QUALIFYING EXAM - FALL 2005

September 8, 2005

INSTRUCTIONS: The exam consists of three parts, each with six problems. Do at least eight problems, including at least two problems from each part.

PART I

1) Let X, Y be spaces, $f : X \to Y$ a continuous bijection.

   (i) Prove that if Y is Hausdorff, then X is also Hausdorff.

   (ii) If X is Hausdorff, must Y also be Hausdorff? Explain.

2) Let A, B be subsets of a space X and denote by $Int(A)$ the interior of A, etc.

   (i) Prove that $Int(A \cap B) = Int(A) \cap Int(B)$.

   (ii) Is $Int(A \cup B)$ necessarily equal to $Int(A) \cup Int(B)$? Explain.

3) If X is a metric space and A is a countable, dense subset of X, prove that X has a countable basis of open sets.

4) State and prove one of the following theorems.

   (i) Tychonoff's Product Theorem for two factors

   (ii) Tietze Extension Theorem

5) Let X, Y be spaces, $f : X \to Y$ a continuous surjection.

(i) Prove that if X is connected, then Y is also connected.

(ii) If Y is connected, must X also be connected? Explain.

6) Let $\{X_i\}$ be a finite collection of locally path-connected spaces.

(i) Prove that the Cartesian product ΠX_i is locally path-connected.

(ii) Is the result in (i) necessarily true if the collection $\{X_i\}$ is infinite? Explain.

PART II

1) Let X, Y be spaces with basepoints $x \in X$, $y \in Y$. Prove that $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.

2) Prove that the Euclidean spaces $\mathbb{R}^2$, $\mathbb{R}^3$ are not homeomorphic.

3) Describe all the covering spaces of $S^1 \times \mathbb{R}P^3$.

4) Use a covering space argument to prove that any map $\mathbb{R}P^2 \times \mathbb{R}P^2 \to S^1$ is null-homotopic.

5) (i) State the Seifert - van Kampen Theorem.

(ii) Using this theorem, or otherwise, construct a space with fundamental group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

6) Is $\mathbb{R}P^4 \vee \mathbb{R}P^5$ a retract of $\mathbb{R}P^4 \times \mathbb{R}P^5$? Explain.

PART III

1) Determine the singular chain complex of a singleton space $\{P\}$ and then compute the singular homology groups of $\{P\}$.

2) (i) If $\{X_i\}$ denotes the set of path-components of a space X, state a theorem relating the singular homology group $H_k(X)$ with the set of singular homology groups $\{H_k(X_i)\}$. Use this result to compute all the singular homology groups $H_k(C)$, where C is the Cantor set.

(ii) Compute all the singular homology groups $H_k(EC)$, where EC is the suspension of the Cantor set, and determine whether EC is simply connected.

3) Construct a CW-complex X with $H_0(X) = \mathbb{Z}$, $H_3(X) = \mathbb{Z}/6\mathbb{Z}$, $H_6(X) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_k(X) = 0$ for all $k \neq 0, 3, 6$.

4) Prove that there are no maps $j : \mathbb{C}P^2 \to \mathbb{R}P^4$, $r : \mathbb{R}P^4 \to \mathbb{C}P^2$ such that the composition $r \circ j$ is homotopic to the identity map from $\mathbb{C}P^2$ to itself.

5) If K is the Klein Bottle, compute $H_k(K;G)$ for all k, where $G = \mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$.

6) Prove that $\mathbb{C}P^4$ is not homeomorphic to $S^2 \times S^6$.