Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. Justify your answers and include the names or the precise statements of theorems that you cite.

Part I

1. A map $f : X \to Y$ is proper if and only if the preimage of compact sets is compact. Prove that a space $X$ is Hausdorff if and only if the diagonal map $X \to X \times X$ is proper.

2. Prove that the product of an arbitrary collection of connected spaces is connected.

3. Prove that the following two descriptions of the torus are homeomorphic:
   
   (a) $S^1 \times S^1$
   
   (b) A square with opposite sides identified as shown.

Part II

4. The map $p : C \to G$ defined by $u_i w_j \mapsto v_i v_j$ defines a covering map from the graph $C$ on the left to the graph $G$ on the right.

   \begin{center}
   \begin{tikzpicture}
   \vertex at (0,0) (u1) [label=below:$u_1$];
   \vertex at (1,0) (u2) [label=below:$u_2$];
   \vertex at (2,0) (u3) [label=below:$u_3$];
   \vertex at (3,0) (w1) [label=below:$w_1$];
   \vertex at (4,0) (w2) [label=below:$w_2$];
   \vertex at (5,0) (w3) [label=below:$w_3$];
   \vertex at (6,0) (w4) [label=below:$w_4$];
   \vertex at (7,0) (w5) [label=below:$w_5$];
   \vertex at (8,0) (v1) [label=below:$v_1$];
   \vertex at (9,0) (v2) [label=below:$v_2$];
   \vertex at (10,0) (v3) [label=below:$v_3$];
   \vertex at (11,0) (v4) [label=below:$v_4$];
   \vertex at (12,0) (v5) [label=below:$v_5$];
   \draw (u1) to (u2);
   \draw (u2) to (u3);
   \draw (u3) to (u1);
   \draw (w1) to (w2);
   \draw (w2) to (w3);
   \draw (w3) to (w4);
   \draw (w4) to (w5);
   \draw (w5) to (w1);
   \draw (v1) to (v2);
   \draw (v2) to (v3);
   \draw (v3) to (v4);
   \draw (v4) to (v5);
   \draw (v5) to (v1);
   \end{tikzpicture}
   \end{center}

   Describe the automorphism group $\text{aut}(p)$ of this cover and the image of the fundamental group under the map $p_* : \pi_1(C,u_1) \to \pi_1(G,v_1)$.

5. A topological group is a topological space $G$ together with continuous product and inverse maps, $G \times G \to G$ and $G \to G$ that make the $G$ into a group. Prove that if $G$ is a topological group and $e \in G$ is the identity, then $\pi_1(G,e)$ is abelian.

6. Let $X = S^1 \cup_f D^2$ where $f(z) = z^5$.
   
   (a) Compute $\pi_1(X)$.
   
   (b) Prove that any map $g : X \to S^1$ is null-homotopic.
7. Let $T = \mathbb{R}^2 / \mathbb{Z}^2$ be the 2-torus.
   (a) Let $L$ be a line of non-zero rational slope $p/q$ in $\mathbb{R}^2$. Prove that $L$ projects to a homotopically non-trivial curve $\alpha_{p,q}$ in $T$.
   (b) Let $X = T \cup_f D^2$, where $f : \partial D^2 \to \alpha_{p,q}$ identifies $S^1$ to $\alpha_{p,q}$. Compute $\pi_1(X)$.

8. Let $X$ be the surface as shown.
   (a) Compute $\pi_1(X)$.
   (b) Identify the surface using the classification of surfaces.

Part III

9. The suspension $\Sigma X$ of $X$ is the quotient space
   \[ \Sigma X = (X \times [0, 1]) / \{(x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1) \text{ for all } x_1, x_2 \in X\}. \]
   (a) Prove that $\tilde{H}_{n+1}(\Sigma X) \cong \tilde{H}_n(X)$.
   (b) Compare $H_1(\Sigma X)$ and $H_0(X)$.

10. Let $X$ be the subset of $\mathbb{R}^3$ consisting of the union of the spheres $A = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3; (x - 1)^2 + y^2 + z^2 = 1\}$. Compute $H_*(X)$.

11. Let $T^n = (S^1)^n$ be the n-torus. Compute $\chi(T^n)$.

12. Let $T$ be the torus $S^1 \times S^1$, and let $X$ be the one point union of two copies of $S^1$ and $S^2$, i.e. $S^1 \vee S^1 \vee S^2$. Show that $T$ and $X$ have the same homology groups. Describe the cup product ring structure in each case and deduce that they are not homotopy equivalent.

13. Define the Lefschetz number $\Lambda_f$ of a map $f : X \to X$ to be
   \[ \Lambda_f := \sum_{k \geq 0} (-1)^k \text{trace} \left( H_k(X, \mathbb{Q}) \xrightarrow{f_*} H_k(X, \mathbb{Q}) \right). \]
   Compute the Lefschetz number of the map $f : T^2 \to T^2$ obtained by flipping the 2-dimensional torus upside down.