Topological Qualifying Exam
Mathematics Program, CUNY Graduate Center
Fall 2020

Instructions: Do 7 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 7 problems, identify which 7 should be graded. Justify your answers and clearly indicate which “well-known” theorems you cite.

Part I

1. Show that if $X$ is Hausdorff and locally compact then the one-point compactification of $X$ is Hausdorff.

2. Prove that if $\{X_i | i \in I\}$ is a family of connected spaces such that $\bigcap_{i \in I} X_i \neq \emptyset$, then $\bigcup_{i \in I} X_i$ is connected.

3. Let $X = \mathbb{R}$ with the basis for its topology all open intervals $(a, b)$ for $a < b$ in $\mathbb{R}$.
   
   Let $Y = \mathbb{R}$ with the basis for its topology all closed-open intervals $[a, b)$ for $a < b$ in $\mathbb{R}$.
   
   Let $f: X \to Y$ be given by $x \mapsto x$. Let $g: Y \to X$ be given by $x \mapsto x$.
   
   Justify for each of $f$ and $g$, whether these functions are continuous, open, and/or closed.
   
   (Here, closed means the image of each closed set is closed, and likewise for open).

4. (a) Let $X$ be a Hausdorff topological space. If $\{x_n\}$ is a convergent sequence in $X$, prove that $\lim_{n \to \infty} x_n$ is unique.
   
   (b) Suppose $f: X \to Y$ is a continuous surjective function. Show that if $X$ is compact and $Y$ is Hausdorff, then $f$ is a quotient map.

Part II

5. Let $X$ be obtained from the torus $T^2$ by removing a small open disk, and identifying the antipodal points of the resulting boundary circle on $T^2$ (see figure).
   
   (a) Use van Kampen’s Theorem to write down a presentation for $\pi_1(X)$.
   
   (b) Compute the homology $H_*(X)$ using a $\Delta$-complex structure. Verify that your answer agrees with part (a).

6. For $n \geq 2$, let $X_n$ be the quotient of $n$ 2-disks $\{D^2_1, \ldots, D^2_n\}$ with their boundary circles identified. Let $Y_n = S^1 \sqcup fD^2$, where $f(z) = z^n$.
   
   (a) Prove that $\pi_1(Y_n) = \mathbb{Z}/n\mathbb{Z}$, and that $X_n$ is the universal covering space for $Y_n$.
   
   (b) Use $X_6$ to describe all isomorphism classes of path-connected covering spaces of $Y_6$.

7. Let $X$ be a CW-complex such that $H_1(X) = \mathbb{Z}/3$. Let $T^3$ be the 3-torus. Prove that every continuous map $f: X \to T^3$ is homotopic to a constant map.

8. Describe three connected non-homeomorphic 2-fold covering spaces of $\mathbb{R}P^2 \vee S^1$.
   
   (a) Justify algebraically.
   
   (b) Sketch the covers.
Part III

9. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same homology groups in all dimensions. Are they homotopy equivalent? Justify.

10. Let $X$ be the space consisting of a solid torus, with an open neighbourhood of a curve running twice around its interior removed, as illustrated below (glue the left end to the right end by the identity map).

Use the Mayer-Vietoris Theorem to compute the homology groups of $X$.

11. (a) Compute the reduced homology group $\tilde{H}_n(\mathbb{RP}^n)$ for all $n$. Justify.

(b) Use the long exact sequence of a pair to compute $\tilde{H}_{n-1}(\mathbb{RP}^n)$ for all $n$. Justify.

12. Let $m, n \geq 1$.

(a) Describe the cohomology rings $H^*(\mathbb{RP}^m \vee \mathbb{RP}^n; \mathbb{Z}/2)$ and $H^*(\mathbb{RP}^m \times \mathbb{RP}^n; \mathbb{Z}/2)$.

(b) Show that $\mathbb{RP}^m \vee \mathbb{RP}^n$ cannot be a retract of $\mathbb{RP}^m \times \mathbb{RP}^n$.

13. Let $T$ denote the torus and $K$ denote the Klein bottle.

(a) Prove that for any map $f : T \to K$, the map $f^* : H^2(K; \mathbb{Z}_2) \to H^2(T; \mathbb{Z}_2)$ is trivial.

(b) Using the cup product on $H^*(T)$, show that for any non-zero $\alpha \in H^1(T)$, there exists $\beta \in H^1(T)$ such that $\alpha \cup \beta \neq 0$. 

2